Lecture 4 Extra Problems

Pointed Curves, Compact Type, Jacobians, and the Boundary of $\overline{\mathcal{M}}_g$

Throughout, work over an algebraically closed field of characteristic 0 unless otherwise stated. These problems extend the discussion of stable curves and $\overline{\mathcal{M}}_g$ to pointed curves, compact-type curves, generalized Jacobians, and the boundary of the moduli stack.

Problem 1. Why $\overline{\mathcal{M}}_{0,n}$ is a fine moduli space

Let $g, n \geq 0$ with $2g - 2 + n > 0$. An *n-pointed stable curve of genus g* is a connected proper nodal curve

$$(C; p_1, \ldots, p_n)$$

of arithmetic genus g, together with n distinct smooth marked points, such that

$$\omega_C(p_1 + \cdots + p_n)$$

is ample. We take for granted that this is equivalent to requiring, for every irreducible component E of the normalization,

$$2g(E) - 2 + n_E > 0,$$

where n_E is the number of special points on E (preimages of nodes together with marked points). The moduli stack of stable n-pointed genus-g curves is denoted

$$\overline{\mathcal{M}}_{g,n}.$$

Now specialize to $g = 0$ and assume $n \geq 3$.

(a) Show that every irreducible component of a stable n-pointed genus-0 curve is isomorphic to $\mathbb{P}^1$ and contains at least three special points.

(b) Prove that any automorphism of a stable pointed genus-0 curve fixing all marked points is the identity.

Hint. First show that an automorphism of the dual tree fixing all labeled legs is the identity. Then use that an automorphism of $\mathbb{P}^1$ fixing three points is the identity.

(c) Deduce that every object of $\overline{\mathcal{M}}_{0,n}$ has trivial automorphism group. Explain why $\overline{\mathcal{M}}_{0,n}$ is therefore represented by a scheme, customarily denoted

$$\overline{M}_{0,n},$$

and why this is a fine moduli space carrying a universal stable n-pointed rational curve.

(d) Determine $\overline{M}_{0,3}$.

(e) Show, using the cross-ratio, that

$$M_{0,4} \simeq \mathbb{P}^1 \setminus \{0, 1, \infty\}$$

and that

$$\overline{M}_{0,4} \simeq \mathbb{P}^1.$$

(f) Interpret the three boundary points $0, 1, \infty$ as the three ways to split four labeled marked points into two pairs on a nodal stable curve.

Problem 2. Curves of compact type

Let C be a stable nodal curve, with dual graph Γ_C. We say that C is of *compact type* if Γ_C is a tree. Let

$$\mathcal{M}_g^{\mathrm{ct}} \subset \overline{\mathcal{M}}_g$$

denote the compact-type locus.

(a) Show that the following conditions are equivalent:

(i) Γ_C is a tree;

(ii) every node of C is separating;

(iii) removing any node disconnects C.

(b) If $C_1, \ldots, C_r$ are the irreducible components of a compact-type curve, show that

$$g(C) = \sum_{i=1}^{r} g(C_i).$$

(c) Give one stable genus-2 curve of compact type and one stable genus-2 curve that is not of compact type.

(d) Explain why every smooth curve is of compact type, so that

$$\mathcal{M}_g \subset \mathcal{M}_g^{\mathrm{ct}} \subset \overline{\mathcal{M}}_g.$$

(e) Show that compact type is an open condition in families. Conclude that

$$\mathcal{M}_g^{\mathrm{ct}} \subset \overline{\mathcal{M}}_g$$

is an open substack.

Problem 3. Jacobians of compact-type curves

Let

$$C = C_1 \cup \cdots \cup C_r$$

be a stable curve of compact type.

(a) Let L be a line bundle of multidegree 0 on C. Show that L is determined, up to isomorphism, by its restrictions

$$L_i = L|_{C_i}.$$

Explain why the tree condition eliminates any nontrivial gluing parameter around a cycle.

(b) Deduce that

$$\mathrm{Pic}^0(C) \simeq \prod_{i=1}^{r} \mathrm{Pic}^0(C_i).$$

(c) Conclude that the Jacobian of a compact-type curve is an abelian variety and that

$$J(C) \simeq \prod_{i=1}^{r} J(C_i).$$

(d) Verify that

$$\dim J(C) = \sum_i g(C_i) = g(C).$$

(e) Let

$$C = E_1 \cup_p E_2$$

be a stable genus-2 curve consisting of two elliptic curves meeting at one point. Show that

$$J(C) \simeq E_1 \times E_2$$

with the product principal polarization.

(f) Explain why this gives a geometric reason for the terminology *compact type*.

Problem 4. Generalized Jacobians away from compact type

Let C be an irreducible curve with one node, and let

$$\nu : \widetilde{C} \longrightarrow C$$

be its normalization. Let $p, q \in \widetilde{C}$ be the two preimages of the node.

(a) Show that a degree-zero line bundle on C can be described by a degree-zero line bundle L on $\widetilde{C}$, together with an identification

$$L_p \simeq L_q.$$

(b) Show that, after fixing L, the gluing identification is parametrized by $\mathbb{G}_m$.

(c) Deduce an exact sequence

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \operatorname{Pic}^0(C) \longrightarrow \operatorname{Pic}^0(\widetilde{C}) \longrightarrow 0.$$

Conclude that $\operatorname{Pic}^0(C)$ is not proper.

(d) More generally, let C be a connected nodal curve with irreducible components C_i and dual graph Γ. Explain why its generalized Jacobian fits into an exact sequence

$$1 \longrightarrow (\mathbb{G}_m)^{h^1(\Gamma)} \longrightarrow \operatorname{Pic}^0(C) \longrightarrow \prod_i J(C_i) \longrightarrow 0.$$

(e) Deduce the criterion

$$\boxed{C \text{ is of compact type} \quad \Longleftrightarrow \quad \operatorname{Pic}^0(C) \text{ is an abelian variety.}}$$

Problem 5. The boundary divisors of $\overline{\mathcal{M}}_g$

Let $g \geq 2$, and set
$$\Delta = \overline{\mathcal{M}}_g \setminus \mathcal{M}_g.$$

(a) Explain why the generic point of a codimension-one component of Δ corresponds to a stable curve with exactly one node. Why does smoothing one node give one deformation parameter?

(b) Let C be a stable genus-g curve with exactly one node p. Show that there are two possibilities:

(i) p is nonseparating, and the normalization of C is connected of genus $g-1$;

(ii) p is separating, and
$$C = C_1 \cup_p C_2, \qquad g(C_1) = i, \quad g(C_2) = g - i$$
for some
$$1 \leq i \leq \left\lfloor \frac{g}{2} \right\rfloor.$$

(c) Define Δ_0 to be the closure of the locus of irreducible one-nodal curves. Describe a generic member by starting with a smooth genus-$(g-1)$ curve with two distinct points and identifying those points. Relate this construction to the clutching morphism
$$\overline{\mathcal{M}}_{g-1,2} \longrightarrow \overline{\mathcal{M}}_g.$$

(d) For
$$1 \leq i \leq \left\lfloor \frac{g}{2} \right\rfloor,$$
define Δ_i to be the closure of the locus of curves
$$C = C_1 \cup_p C_2, \qquad g(C_1) = i, \quad g(C_2) = g - i.$$
Relate Δ_i to the clutching morphism
$$\overline{\mathcal{M}}_{i,1} \times \overline{\mathcal{M}}_{g-i,1} \longrightarrow \overline{\mathcal{M}}_g.$$

(e) Using
$$\dim \overline{\mathcal{M}}_{g,n} = 3g - 3 + n,$$
verify that each Δ_i has dimension $3g-4$. Conclude that the boundary is the union of the irreducible divisors
$$\boxed{\overline{\mathcal{M}}_g \setminus \mathcal{M}_g = \Delta_0 \cup \Delta_1 \cup \cdots \cup \Delta_{\lfloor g/2 \rfloor}.}$$

(f) Which of the divisors Δ_i parameterize curves of compact type at their generic point? Relate your answer to the separating versus nonseparating nature of the node.

(g) Let $\mathcal{A}_g$ denote the moduli stack of principally polarized abelian varieties of dimension g. Using Problem 3, explain why the Torelli map extends to the compact-type locus:
$$\tau^{\mathrm{ct}} : \mathcal{M}_g^{\mathrm{ct}} \longrightarrow \mathcal{A}_g, \qquad C \longmapsto J(C).$$

For a generic curve

$$C = C_1 \cup_p C_2 \in \Delta_i, \qquad i > 0,$$

show that

$$J(C) \simeq J(C_1) \times J(C_2),$$

so the Torelli image does not remember the attaching point(s). Compare $\dim \Delta_i = 3g - 4$ with the dimension of its image: for $2 \leq i \leq g - i$, show that the image has dimension at most $3g - 6$, while for $i = 1$ and $g \geq 3$ it has dimension at most $3g - 5$. Conclude that for $g \geq 3$ the extended Torelli map is not quasi-finite (and in particular is not injective) along the compact-type boundary. What is exceptional about $g = 2$?