Lecture 1 Problems

Moduli Functors, Grassmannians, and Hilbert Schemes

Throughout, work over $\mathbb{C}$. Unless otherwise stated, schemes are of finite type over $\mathbb{C}$. The goal of these problems is to complement the first lecture; the hints are intentionally brief and are not meant to constitute solutions.

Problem 1. Representable functors are sheaves

Let X be a scheme and let

$$S = \bigcup_i U_i$$

be a Zariski open cover.

(a) Show that giving a morphism $f : S \to X$ is equivalent to giving morphisms

$$f_i : U_i \to X$$

that agree on every overlap $U_i \cap U_j$.

(b) Deduce that every representable functor

$$h_X(S) = \mathrm{Hom}(S, X)$$

is a sheaf for the Zariski topology.

(c) Give an example of a set-valued moduli functor for which the sheaf condition is not automatic from its definition. What information might be lost when one first passes to isomorphism classes?

Hint. For (a), use the fact that morphisms of locally ringed spaces glue. For (c), think about objects that are locally isomorphic but glued by nontrivial automorphisms.

Problem 2. Isotrivial families and representability

Let F be a moduli functor, and suppose that F is represented by a scheme M. Let S be a connected, reduced scheme of finite type over $\mathbb{C}$, and let

$$\mathcal{X} \longrightarrow S$$

be a family such that every closed fiber $\mathcal{X}_s$ is isomorphic to a fixed object X.

(a) Let

$$f : S \longrightarrow M$$

be the classifying morphism. Show that $f(s)$ is the point of M corresponding to X for every closed point $s \in S$.

(b) Prove that if a morphism from a reduced finite-type $\mathbb{C}$-scheme S to a scheme M sends every closed point of S to a single closed point $m \in M$, then the morphism factors through m.

(c) Deduce that the family $\mathcal{X} \to S$ is trivial:

$$\mathcal{X} \simeq X \times S.$$

(d) Conclude that a nontrivial isotrivial family over such a base obstructs representability of the set-valued moduli functor by a scheme.

(e) Explain why reducedness of S matters. What goes wrong for

$$S = \operatorname{Spec} \mathbb{C}[\epsilon]/(\epsilon^2)?$$

Hint. For (b), work locally on M near m, reduce to a statement about regular functions, and use the Nullstellensatz plus reducedness. For (e), a nonreduced scheme may have only one closed point while still carrying nonconstant maps.

Problem 3. The Grassmannian as a moduli space

Fix integers $0 < r < n$. Define $\mathrm{Gr}(r,n)(S)$ to be the set of rank-r subbundles

$$E \subset \mathcal{O}_S^n,$$

where the quotient $\mathcal{O}_S^n/E$ is also locally free.

(a) Why is it too restrictive to require $E \simeq \mathcal{O}_S^r$ globally? Give a natural example of a nontrivial line subbundle of a trivial rank-2 bundle.

(b) Fix $I \subset \{1,\dots,n\}$ with $|I| = r$, and let U_I be the subfunctor on which the projection

$$E \longrightarrow \mathcal{O}_S^I$$

is an isomorphism. Show that this is an open condition on S.

(c) Show that on U_I, the subbundle E is uniquely the graph of a homomorphism

$$\mathcal{O}_S^I \longrightarrow \mathcal{O}_S^{I^c}.$$

Deduce that

$$U_I \simeq \mathbb{A}^{r(n-r)}.$$

(d) Show that the U_I cover the Grassmannian functor and deduce representability of $\mathrm{Gr}(r,n)$ by gluing these affine charts.

(e) Work out $\mathrm{Gr}(1,2)$ explicitly, including the transition map between the two standard affine charts, and recover

$$\mathrm{Gr}(1,2) \simeq \mathbb{P}^1.$$

Hint. For (a), use the tautological bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \subset \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$. For (b)–(d), choose an $r \times r$ minor and invert its determinant; on that open set row-reduce the matrix to a graph form.

Problem 4. Two points collide: Hilbert scheme versus Chow variety

Work first on the affine chart

$$\mathbb{A}^2 = \operatorname{Spec} \mathbb{C}[x, y] \subset \mathbb{P}^2.$$

Let

$$p = (0, 0), \qquad q_t = (t, 0), \qquad t \neq 0,$$

and let $Z_t = \{p, q_t\}$.

(a) Show that

$$I_{Z_t} = (y,\, x(x - t)).$$

Consider the family $\mathcal{Z} \subset \mathbb{A}^2 \times \mathbb{A}^1_t$ defined by

$$I = (y,\, x(x - t)).$$

Show that $\mathcal{Z} \to \mathbb{A}^1_t$ is finite flat of degree 2, and determine its special fiber.

(b) Repeat the construction with $q'_t = (0, t)$. Show that the new special fiber is a different length-2 subscheme supported at p.

(c) Interpret the difference between the two special fibers as a tangent direction at p. Show that the nonreduced length-2 subschemes supported at p are parametrized by

$$\mathbb{P}(T_p\mathbb{P}^2) \simeq \mathbb{P}^1.$$

(d) The Chow variety of degree-2 zero-cycles on $\mathbb{P}^2$ is

$$\mathrm{Chow}_2(\mathbb{P}^2) = \mathrm{Sym}^2(\mathbb{P}^2).$$

What is the limit in $\mathrm{Sym}^2(\mathbb{P}^2)$ of $[p] + [q_t]$ as $t \to 0$? Compare this with the two different limits obtained in the Hilbert scheme.

(e) Let

$$\rho : \mathrm{Hilb}^2(\mathbb{P}^2) \longrightarrow \mathrm{Sym}^2(\mathbb{P}^2)$$

be the Hilbert–Chow morphism. Describe the fibers of ρ over $[p] + [q]$ for $p \neq q$ and over $2[p]$.

(f) Explain, using this example, what flatness accomplishes in the Hilbert functor and what information the Chow variety forgets.

Hint. For (a), compute the quotient ring and look for a free $\mathbb{C}[t]$-basis. For (b), interchange x and y. For (c), a length-2 subscheme supported at a smooth point is determined by a codimension-one subspace of the cotangent space, equivalently a line in the tangent space. For (d), cycles retain multiplicity but not infinitesimal direction.

Problem 5. How much of the ideal determines two points in $\mathbb{P}^2$?

Let $Z \subset \mathbb{P}^2$ be a zero-dimensional subscheme of length 2. Its Hilbert polynomial is $P_Z(m) = 2$.

(a) Suppose first that $Z = \{p, q\}$ with $p \neq q$. Show that

$$h^0(\mathbb{P}^2, \mathcal{I}_Z(1)) = 1.$$

What geometric information is contained in $H^0(\mathcal{I}_Z(1))$? Why does it not determine the pair $\{p, q\}$?

(b) Choose coordinates

$$p = [1 : 0 : 0], \qquad q = [0 : 1 : 0].$$

Show that

$$I_Z = (z, xy).$$

Compute $I_Z(2)$ and show that the degree-2 data determine Z scheme-theoretically.

(c) Now consider the nonreduced length-2 subscheme

$$I_Z = (z, y^2).$$

Compute $I_Z(1)$ and $I_Z(2)$. Which degree first detects the infinitesimal direction?

(d) Show that for every length-2 subscheme $Z \subset \mathbb{P}^2$,

$$h^0(Z, \mathcal{O}_Z(2)) = 2.$$

Since $h^0(\mathbb{P}^2, \mathcal{O}(2)) = 6$, deduce that Z determines a point of

$$\mathrm{Gr}(2, 6).$$

(e) Explain why this example suggests that in constructing a Hilbert scheme one should choose a degree m that is *uniformly sufficiently large* for every subscheme with the prescribed Hilbert polynomial.

Hint. For (a), two distinct points impose two independent conditions on linear forms. For (b), use the displayed ideal rather than reconstructing the points from scratch. For (c), compare the common zero scheme of the degree-1 and degree-2 pieces. For (d), use that twisting a line bundle on a zero-dimensional scheme does not change its length.

Problem 6. Castelnuovo–Mumford regularity and boundedness

Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$. Recall that $\mathcal{F}$ is called *m-regular* if

$$H^i(\mathbb{P}^n, \mathcal{F}(m - i)) = 0 \qquad \text{for every } i > 0.$$

You may use the following standard consequences: if $\mathcal{F}$ is m-regular, then it is m'-regular for all $m' \geq m$, $\mathcal{F}(m)$ is globally generated, and the natural multiplication maps in higher degrees are surjective.

> **You may use the following fact without proof.** For every Hilbert polynomial P, there exists an integer $m_0(P)$ such that for every closed subscheme $Z \subset \mathbb{P}^n$ with Hilbert polynomial P, the ideal sheaf $\mathcal{I}_Z$ is $m_0(P)$-regular. This is the uniform boundedness supplied by Gotzmann regularity.

Fix $m \geq m_0(P)$.

(a) Starting from

$$0 \longrightarrow \mathcal{I}_Z(m) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(m) \longrightarrow \mathcal{O}_Z(m) \longrightarrow 0,$$

show that restriction induces a surjection

$$H^0(\mathbb{P}^n, \mathcal{O}(m)) \twoheadrightarrow H^0(Z, \mathcal{O}_Z(m)),$$

and that the target has dimension $P(m)$.

(b) Deduce that every Z with Hilbert polynomial P determines a point of the finite-dimensional Grassmannian

$$\mathrm{Gr}\Big(P(m), H^0(\mathbb{P}^n, \mathcal{O}(m))\Big).$$

Explain precisely where the uniformity in $m_0(P)$ is used.

(c) Let

$$W = H^0(\mathbb{P}^n, \mathcal{I}_Z(m)) \subset H^0(\mathbb{P}^n, \mathcal{O}(m)).$$

Why does an arbitrary subspace W of the correct dimension need not come from a homogeneous ideal? Identify the multiplication map that records the first compatibility condition.

(d) Explain why suitable rank conditions on multiplication maps define closed conditions on the Grassmannian.

(e) Taking the relevant persistence theorem as a black box, summarize the construction of the Hilbert scheme in the form

$$\text{uniform regularity} \implies \text{Grassmannian} \implies \text{closed conditions} \implies \mathrm{Hilb}^P(\mathbb{P}^n).$$

Why does this also account for projectivity of the Hilbert scheme?

Hint. For (a), regularity gives the needed H^1-vanishing. For (c), multiply degree-m equations by linear forms. For (d), recall that a rank bound is expressed by vanishing of minors. For (e), the persistence theorem is the step that turns finitely many degree conditions into the full Hilbert-polynomial condition.

The exercises are designed to emphasize the logic of representability rather than the technical proofs of boundedness and persistence.