Lecture 2 Problems

Automorphisms, Torsors, and Quotient Stacks

Throughout, work over $\mathbb{C}$ unless otherwise stated. The goal of these problems is to complement the second lecture with concrete examples; the hints are intentionally brief and are not meant to constitute solutions.

Convention. A G-torsor is a right G-torsor. If X is a left G-scheme, an object of $[X/G](S)$ is a right G-torsor $P \to S$ together with a map $f : P \to X$ satisfying

$$f(pg) = g^{-1} f(p).$$

Problem 1. An étale-locally trivial torsor that is not Zariski-locally trivial

Fix an integer $n > 1$, and consider

$$P = \operatorname{Spec} \mathbb{C}[y, y^{-1}] \longrightarrow \mathbb{G}_m = \operatorname{Spec} \mathbb{C}[x, x^{-1}]$$

defined by $x = y^n$. Let μ_n act on P by

$$y \cdot \zeta = \zeta y.$$

(a) Show that $P \to \mathbb{G}_m$ is finite étale of degree n.

(b) Show that $P \to \mathbb{G}_m$ is a μ_n-torsor.

(c) Show that this torsor has no global section.

(d) Show that the restriction of this torsor to every nonempty Zariski open subset

$$U \subset \mathbb{G}_m$$

is nontrivial.

(e) Show that the torsor becomes trivial after an étale base change. Explain why this example shows that the Zariski topology is too coarse for the general theory of G-torsors.

Hint. For (c), a section would produce an n-th root of x in $\mathbb{C}[x, x^{-1}]$. For (d), use irreducibility: $P|_U$ is a nonempty open subset of the irreducible scheme P, whereas a trivial μ_n-torsor $\mu_n \times U$ is a disjoint union of n nonempty copies of U. For (e), pull back along $P \to \mathbb{G}_m$ itself.

Problem 2. A nontrivial family of $\mathbb{P}^1$'s

Let

$$E = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$$

and consider the projective bundle

$$X = \mathbb{P}(E) \longrightarrow \mathbb{P}^1.$$

(a) Show that every fiber of $X \to \mathbb{P}^1$ is isomorphic to $\mathbb{P}^1$.

(b) Show that the family is Zariski locally isomorphic to

$$\mathbb{P}^1 \times U \longrightarrow U.$$

(c) Show that X is not isomorphic, as a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$, to

$$\mathbb{P}^1 \times \mathbb{P}^1.$$

(d) Interpret the transition functions of $X \to \mathbb{P}^1$ as PGL_2-valued transition functions, and explain why the family determines a nontrivial PGL_2-torsor.

Hint. For (b), trivialize E on the two standard affine charts of $\mathbb{P}^1$. For (c), you may identify $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1))$ with the Hirzebruch surface $\mathbb{F}_1$ and use its section of self-intersection -1, or use the classification of projective bundles up to tensoring by a line bundle.

Problem 3. The naive prestack BG^{pre} and failure of descent

Let G be an algebraic group. Define a groupoid-valued pseudofunctor BG^{pre} by declaring that for every scheme S, the groupoid $BG^{\mathrm{pre}}(S)$ has a single object $*_S$ with

$$\mathrm{Aut}(*_S) = G(S).$$

For a morphism $f : T \to S$, pullback sends $*_S$ to $*_T$ and $g \in G(S)$ to its pullback $f^*g \in G(T)$.

(a) Let $\{U_i \to S\}$ be a cover. Show that a descent datum for the objects $*_{U_i}$ is equivalent to a collection

$$g_{ij} \in G(U_i \times_S U_j)$$

satisfying the cocycle condition

$$g_{ij} g_{jk} = g_{ik}$$

on triple overlaps.

(b) Suppose that this descent datum is induced by the unique global object $*_S$. Show that, after choosing isomorphisms between its restrictions and the local objects, the cocycle must be of the form

$$g_{ij} = h_i^{-1} h_j$$

for some $h_i \in G(U_i)$. In particular, an effective descent datum in BG^{pre} must be a coboundary.

(c) Take $G = \mu_n$, $S = \mathbb{G}_m$, and use the étale cover

$$P \to S, \qquad x = y^n,$$

from Problem 1. The corresponding μ_n-torsor determines descent data for the unique object of $BG^{\mathrm{pre}}(P)$. Show that this descent datum is not effective in BG^{pre}.

(d) Conclude that BG^{pre} does not satisfy descent for objects. Explain why enlarging $BG^{\mathrm{pre}}(S)$ to include all G-torsors over S repairs this failure and leads to the classifying stack BG.

Hint. For (b), compare two choices of local trivialization of a putative global object. For (c), if the descent datum were effective, the μ_n-torsor from Problem 1 would be trivial. Thus G-torsors appear precisely as the new objects needed to make descent effective.

Problem 4. Unordered triples of points on $\mathbb{P}^1$

Let

$$U \subset \mathrm{Sym}^3(\mathbb{P}^1)$$

be the open subset parametrizing unordered triples $\{p_1, p_2, p_3\}$ of distinct points. The group PGL_2 acts naturally on U.

(a) Show that PGL_2 acts transitively on U.

(b) Let

$$D = \{0, 1, \infty\}.$$

Determine the stabilizer $\mathrm{Stab}_{\mathrm{PGL}_2}(D)$ by showing that every permutation of $\{0, 1, \infty\}$ is induced by a unique element of PGL_2. Deduce that

$$\mathrm{Stab}_{\mathrm{PGL}_2}(D) \simeq S_3.$$

(c) Show that

$$[U/\,\mathrm{PGL}_2] \simeq BS_3,$$

and interpret this as a moduli statement about unordered triples of distinct points on $\mathbb{P}^1$.

(d) How does the answer change if the three points are ordered? In particular, compute the stabilizer of $(0, 1, \infty)$ as an ordered triple.

Hint. Use the simple transitivity of PGL_2 on ordered triples of distinct points. For (b), transformations such as $z \mapsto 1 - z$ and $z \mapsto 1/z$ generate the permutations. For (c), identify U with the homogeneous space PGL_2 / S_3 and use the general principle $[G/H \,/\, G] \simeq BH$.

Problem 5. A weighted projective line as an orbifold

Let $\mathbb{G}_m$ act on $\mathbb{A}^2 \setminus \{0\}$ with weights 1 and 2:

$$t \cdot (x, y) = (tx, t^2 y).$$

Define the weighted projective stack

$$\mathcal{P}(1, 2) = \big[(\mathbb{A}^2 \setminus \{0\})/\mathbb{G}_m\big].$$

(a) Determine the stabilizer of a geometric point (x, y) with $x \neq 0$.

(b) Determine the stabilizer of the point $(0, 1)$ and show that it is μ_2.

(c) Show that the ordinary coarse quotient is $\mathbb{P}^1$, and explain why $\mathcal{P}(1, 2)$ can be regarded as $\mathbb{P}^1$ with one stacky point of order 2.

Hint. For (a)–(b), solve the equations $tx = x$ and $t^2 y = y$. For (c), on $x \neq 0$ use the invariant coordinate y/x^2, and on $y \neq 0$ use x^2/y; compare the two affine charts and keep track of the stabilizer at the exceptional orbit.

Problem 6. What does $[\mathbb{A}^1/\mathbb{G}_m]$ classify?

Let $\mathbb{G}_m$ act on $\mathbb{A}^1$ by scalar multiplication.

(a) Show that an object of
$$[\mathbb{A}^1/\mathbb{G}_m](S)$$
is equivalent to a pair (L, s), where L is a line bundle on S and
$$s \in H^0(S, L)$$
is a section.

(b) Describe the isomorphisms
$$(L, s) \longrightarrow (L', s').$$

(c) Let k be algebraically closed. Determine the isomorphism classes of objects of
$$[\mathbb{A}^1/\mathbb{G}_m](k).$$

(d) Compute the automorphism group of each isomorphism class.

Hint. For (a), from a right $\mathbb{G}_m$-torsor $P \to S$ form the associated line bundle
$$L = P \times^{\mathbb{G}_m} \mathbb{A}^1.$$
A section corresponds to a function $f : P \to \mathbb{A}^1$ satisfying $f(pg) = g^{-1} f(p)$. For (c)–(d), over a point every line bundle is trivial, so reduce to the action of $k^\times$ on k.