Lecture 3 Problems

Algebraic and Deligne–Mumford Stacks, and Moduli of Curves

Throughout, work over an algebraically closed field of characteristic 0 unless otherwise stated. The problems mix general properties of algebraic and Deligne–Mumford stacks with concrete quotient-stack examples and examples from the moduli of curves. Hints are intentionally brief.

Problem 1. When is BG Deligne–Mumford?

Let G be an algebraic group over a field k. Consider the morphism

$$\operatorname{Spec} k \longrightarrow BG$$

corresponding to the trivial G-torsor.

(a) Let $S \to BG$ correspond to a G-torsor $P \to S$. Show that

$$S \times_{BG} \operatorname{Spec} k \simeq P.$$

(b) Assume now that G is smooth over k. Deduce that

$$\operatorname{Spec} k \longrightarrow BG$$

is smooth and surjective, and show that this atlas is *étale* if and only if G is étale over k.

(c) In characteristic 0, determine which of the following classifying stacks are Deligne–Mumford:

$$B\mu_n, \qquad B\mathbb{G}_m, \qquad BGL_n.$$

Interpret the answer in terms of automorphism groups of geometric objects.

(d) Now assume $\operatorname{char} k = p > 0$. Consider the μ_p-torsor

$$\mathbb{G}_m \longrightarrow \mathbb{G}_m, \qquad t \longmapsto t^p.$$

It determines a morphism

$$\mathbb{G}_m \longrightarrow B\mu_p.$$

Show that this morphism is a smooth surjective atlas, and conclude that $B\mu_p$ is an algebraic stack. Show that it is not Deligne–Mumford.

Hint. For (a), a T-point of $S \times_{BG} \operatorname{Spec} k$ is a map $T \to S$ together with a trivialization of the pulled-back torsor P_T. For a right G-torsor, choosing such a trivialization is equivalent to choosing a point $p \in P(T)$: send (g, t) to $p(t)g$.

Problem 2. A smooth stack with a singular coarse quotient

Let μ_2 act on

$$\mathbb{A}^2 = \operatorname{Spec} k[x, y]$$

by

$$(x, y) \longmapsto (-x, -y),$$

where $\operatorname{char} k \neq 2$.

(a) Determine the stabilizer of every geometric point of $\mathbb{A}^2$.

(b) Show that the quotient stack

$$[\mathbb{A}^2/\mu_2]$$

is a smooth Deligne–Mumford stack.

(c) Compute the invariant ring $k[x, y]^{\mu_2}$ and show that the coarse quotient is

$$\operatorname{Spec} k[u, v, w]/(uw - v^2).$$

(d) Show that this coarse quotient is singular at the origin.

(e) Explain why the quotient stack is smooth even though its coarse quotient is singular. What information is retained by the stack at the origin?

Hint. The invariant ring is generated by x^2, xy, y^2. For the singularity, use the Jacobian criterion. The stack has an étale atlas $\mathbb{A}^2 \to [\mathbb{A}^2/\mu_2]$.

Problem 3. Automorphisms of curves of genus $g \geq 2$

Let C be a smooth projective curve over $\mathbb{C}$ of genus $g \geq 2$.

(a) Show that

$$\deg T_C = 2 - 2g < 0,$$

and deduce that

$$H^0(C, T_C) = 0.$$

(b) Use the standard identification

$$T_e \operatorname{Aut}(C) \simeq H^0(C, T_C)$$

to explain why C has no infinitesimal automorphisms.

(c) Choose $m \geq 3$ so that $\omega_C^{\otimes m}$ is very ample. Show that every automorphism of C induces a projective transformation preserving the m-canonical image

$$C \hookrightarrow \mathbb{P}H^0(C, \omega_C^{\otimes m})^\vee.$$

(d) Explain why $\operatorname{Aut}(C)$ is an algebraic group of finite type with zero-dimensional identity component, and conclude that $\operatorname{Aut}(C)$ is finite.

(e) Relate this conclusion to the fact that the moduli stack $\mathcal{M}_g$ is Deligne–Mumford.

Hint. A line bundle of negative degree on a smooth projective curve has no nonzero global sections. For (d), use the pluricanonical embedding to realize $\mathrm{Aut}(C)$ as a closed subgroup of a projective linear group.

Problem 4. The moduli stack $\mathcal{M}_{1,1}$ as a quotient stack

Assume $\operatorname{char} k \neq 2, 3$. Every elliptic curve with chosen origin can be written as a smooth projective Weierstrass cubic

$$E_{a,b} : \qquad y^2 z = x^3 + axz^2 + bz^3 \subset \mathbb{P}^2,$$

with origin $[0:1:0]$ and

$$\Delta(a,b) = -16(4a^3 + 27b^2) \neq 0.$$

Let

$$V = \mathbb{A}^2_{a,b} \setminus V(\Delta).$$

(a) Consider the projective change of coordinates

$$[x:y:z] = [u^2 x' : u^3 y' : z'], \qquad u \in k^\times.$$

Show that it transforms the Weierstrass equation into one with coefficients

$$a' = u^{-4}a, \qquad b' = u^{-6}b.$$

Equivalently, after replacing u by u^{-1}, the natural $\mathbb{G}_m$-action on the coefficients is

$$u \cdot (a,b) = (u^4 a, u^6 b).$$

(b) Show that two points of $V(k)$ lie in the same $\mathbb{G}_m$-orbit if and only if the corresponding pointed elliptic curves are isomorphic. Explain why this leads to

$$\mathcal{M}_{1,1} \simeq [V/\mathbb{G}_m].$$

(c) Recall

$$j = 1728 \, \frac{4a^3}{4a^3 + 27b^2}.$$

Show that j is invariant under the weighted $\mathbb{G}_m$-action. Explain why the coarse moduli space of $\mathcal{M}_{1,1}$ is the affine j-line.

(d) Determine the stabilizer of (a,b) in the three cases

$$ab \neq 0, \qquad a = 0, \qquad b = 0.$$

Deduce that the generic stabilizer is μ_2, and identify the larger stabilizers at the special j-invariants 0 and 1728.

Hint. For (c), the numerator and denominator of j have the same weight 12. For (d), solve $u^4 a = a$ and $u^6 b = b$. When $ab \neq 0$ this gives $u^2 = 1$; when one coefficient vanishes the stabilizer becomes larger.

Problem 5. The hyperelliptic locus is stacky

Let C be a hyperelliptic smooth curve of genus $g \geq 2$, with a degree-two map

$$C \longrightarrow \mathbb{P}^1.$$

(a) Show that the nontrivial deck transformation defines the hyperelliptic involution

$$\iota : C \longrightarrow C,$$

and deduce that

$$\mathbb{Z}/2\mathbb{Z} \subseteq \mathrm{Aut}(C).$$

Explain how this subgroup appears in the stabilizer of the geometric point $[C] \in \mathcal{M}_g$.

(b) Explain what information about C is lost when one passes from the stack $\mathcal{M}_g$ to the coarse moduli space M_g.

(c) Consider the genus-two curve

$$C : \qquad y^2 = x^6 - 1.$$

Show that $x \mapsto \zeta_6 x$ gives an automorphism in addition to the hyperelliptic involution. Deduce that some points of $\mathcal{M}_2$ have stabilizer strictly larger than the generic hyperelliptic $\mathbb{Z}/2\mathbb{Z}$. Find as much of $\mathrm{Aut}(C)$ as you can.

(d) Use the fact that every smooth genus-two curve is hyperelliptic to show that every geometric point of $\mathcal{M}_2$ has at least the hyperelliptic involution in its stabilizer, and explain why this gives a concrete reason that $\mathcal{M}_2$ should be regarded as a stack rather than a scheme.

Hint. For (a), the degree-two extension of function fields is Galois with group $\mathbb{Z}/2\mathbb{Z}$. For (c), take a primitive sixth root of unity ζ_6 and check that $x^6 - 1$ is unchanged by $x \mapsto \zeta_6 x$.

Problem 6. The algebraic-space quotient behind a Hopf surface

Let

$$U = \mathbb{A}^2_{\mathbb{C}} \setminus \{0\},$$

and let $\Gamma = \mathbb{Z}$ act on U by

$$n \cdot (x, y) = (2^n x, 2^n y).$$

Regard Γ as the constant étale group scheme, and consider

$$\mathcal{H} = [U/\Gamma].$$

(a) Show that the action of Γ on U is free.

(b) Show that $U \to \mathcal{H}$ is an étale surjective atlas and that every geometric point of $\mathcal{H}$ has trivial stabilizer. Deduce that $\mathcal{H}$ is an algebraic space.

(c) Let

$$L = \{(x, 0) : x \neq 0\} \subset U.$$

Then $L \simeq \mathbb{G}_m$ is Γ-invariant. Let $Y = L/\Gamma$ be the corresponding quotient algebraic space. Show that the induced map $Y \to \mathcal{H}$ is a closed immersion, and show that any nonempty open neighborhood in Y of the image of $1 \in \mathbb{G}_m$ must be all of Y.

(d) Show that

$$\Gamma(Y, \mathcal{O}_Y) = \mathbb{C}[x, x^{-1}]^{\Gamma} = \mathbb{C}.$$

Use this and part (c) to show that Y is not a scheme. Deduce that $\mathcal{H}$ is not a scheme.

(e) In the analytic category, the same action is properly discontinuous. Identify

$$(\mathbb{C}^2 \setminus \{0\})/2^{\mathbb{Z}}$$

as a primary Hopf surface. Contrast the analytic quotient with the algebraic quotient above.

Hint. For (c), the inverse image of an open set in Y is a Γ-invariant open subset of $\mathbb{G}_m$. A proper closed subset of $\mathbb{G}_m$ is finite, while every nonempty Γ-orbit is infinite. For (d), if Y were a scheme, part (c) would force an affine neighborhood of the image of 1 to equal all of Y; an affine scheme with ring of global functions $\mathbb{C}$ would be a point.