Lecture 4 Problems

Stable Curves, Stable Reduction, and Properness

Throughout, work over an algebraically closed field of characteristic 0 unless otherwise stated. The problems mix combinatorial examples of stable curves, local calculations in semistable reduction, and applications of the valuative criterion.

Problem 1. Graph curves and genus 2

A *graph curve* will mean a connected stable nodal curve C such that every irreducible component of its normalization is $\mathbb{P}^1$ and every component has exactly three special points. Equivalently, its dual graph Γ is connected and trivalent; loops and multiple edges are allowed, and a loop contributes 2 to the valence of its vertex.

Let V and E denote the numbers of vertices and edges of Γ.

(a) Using the normalization exact sequence, show that

$$p_a(C) = h^1(\Gamma) = E - V + 1.$$

(b) Using trivalence,

$$3V = 2E,$$

show that a genus-g graph curve has

$$V = 2g - 2, \qquad E = 3g - 3.$$

Interpret these as the numbers of irreducible components and nodes of C.

(c) Specialize to $g = 2$. Show that

$$V = 2, \qquad E = 3,$$

and classify all connected trivalent multigraphs with two vertices.

(d) Show that there are exactly two possibilities:

(i) the *theta graph*, with three edges joining the two vertices;

(ii) the *dumbbell graph*, with one edge joining the two vertices and one loop at each vertex.

Describe the corresponding nodal curves geometrically.

(e) Verify directly in both cases that the arithmetic genus is 2 and that each rational component has exactly three special points.

(f) Determine the automorphism group of each of the two genus-2 graph curves. Relate automorphisms of the curve to automorphisms of the dual graph and to automorphisms of $\mathbb{P}^1$ preserving three special points.

Problem 2. Multiplicities at a node and finite base change

Let $R = \mathbb{C}[[t]]$. Suppose a model of a family of curves is locally given near a point of the special fiber by

$$t = u^m v^n,$$

where $m, n \geq 1$.

(a) Show that the special fiber has two components

$$U = (u = 0), \qquad V = (v = 0)$$

meeting transversely, with multiplicities m and n respectively.

(b) Explain why this family is semistable at the point only when $m = n = 1$.

(c) Let

$$N = \operatorname{lcm}(m, n)$$

and make the finite base change $t = s^N$. Write $d = \gcd(m, n)$, $m = dm'$, $n = dn'$ with $\gcd(m', n') = 1$. Show that

$$u^m v^n = s^N$$

factors into d branches of the form

$$u^{m'} v^{n'} = \zeta s^{m'n'}, \qquad \zeta^d = 1.$$

(d) Describe the normalization of one branch explicitly. After rescaling s by a constant, show that the normalization of

$$k[u, v, s]/(u^{m'} v^{n'} - s^{m'n'})$$

is $k[a, b]$, via

$$u = a^{n'}, \qquad v = b^{m'}, \qquad s = ab.$$

Conclude that the normalized special fiber is reduced and nodal.

(e) Work out the case $m = n = 2$ explicitly. After the base change $t = s^2$, show that

$$u^2 v^2 = s^2$$

splits into the two branches

$$uv = s, \qquad uv = -s.$$

(f) Explain what this calculation says about the role of finite base change and normalization in semistable reduction. Why can further resolution still be necessary in a general model, even after multiplicities have been removed?

Problem 3. Resolving the degeneration $xy = t^2$

Consider

$$X = \operatorname{Spec} \frac{\mathbb{C}[x, y, t]}{(xy - t^2)} \longrightarrow \mathbb{A}^1_t.$$

(a) Show that the fiber over $t \neq 0$ is smooth and that the special fiber is the nodal curve

$$xy = 0.$$

(b) Show that the total space X is singular at the origin. Resolve this A_1 surface singularity by blowing up the origin, and show that the exceptional divisor is a rational curve

$$E \simeq \mathbb{P}^1$$

with

$$E^2 = -2.$$

(c) Describe the special fiber of the resolved family. Show that the strict transforms of the two original components are separated by E:

$$C_1 - E - C_2.$$

(d) Show that E has two special points and compute

$$\deg_E \omega = 0.$$

Explain why the resolved family is semistable but not stable.

(e) Explain geometrically what happens when one passes from this semistable model to the stable model.

Problem 4. A doubled $\mathbb{P}^1$ from a hyperelliptic quotient

Let C be a smooth hyperelliptic curve of genus $g \geq 2$, written as a double cover

$$q : C \longrightarrow \mathbb{P}^1$$

with hyperelliptic involution ι. Let $L = \mathcal{O}_{\mathbb{P}^1}(g + 1)$, so that C may be written as

$$y^2 = f$$

for a section $f \in H^0(\mathbb{P}^1, L^{\otimes 2})$ with simple zeros.

Start from the trivial family

$$C \times \mathbb{A}^1_s \longrightarrow \mathbb{A}^1_s$$

and let μ_2 act by

$$(x, s) \longmapsto (\iota(x), -s).$$

Let

$$X = (C \times \mathbb{A}^1_s)/\mu_2.$$

(a) Show that the invariant coordinate on the base is

$$t = s^2,$$

so that the quotient carries a natural morphism

$$X \longrightarrow \mathbb{A}^1_t.$$

(b) In the double-cover description, set

$$z = sy.$$

Show that z is invariant and that the quotient family is described by

$$\boxed{z^2 = tf.}$$

(c) Show that for $t \neq 0$ the geometric fiber is isomorphic to C.

(d) Show that the central fiber is

$$z^2 = 0,$$

a nonreduced double structure supported on the zero section $\mathbb{P}^1$. In particular,

$$(X_0)_{\text{red}} \simeq \mathbb{P}^1.$$

(e) Show that there is an exact sequence

$$0 \longrightarrow L^{-1} \longrightarrow \mathcal{O}_{X_0} \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow 0.$$

Deduce that

$$p_a(X_0) = g.$$

(f) Make the finite base change $t = s^2$ and normalize the pullback. Show that one recovers the original trivial family

$$C \times \mathbb{A}^1_s \longrightarrow \mathbb{A}^1_s.$$

(g) Explain how this example illustrates why finite base change is needed in the valuative criterion for properness of Deligne–Mumford stacks: after the finite extension $t = s^2$ and normalization, the family acquires an ordinary smooth extension.

Problem 5. Cuspidal curves and failure of separatedness

Let $R = \mathbb{C}[[t]]$. Choose distinct nonzero numbers

$$a_1, a_2, a_3, \qquad b_1, b_2, b_3.$$

For $t \neq 0$, let C_t be the double cover of $\mathbb{P}^1$ branched over the six points

$$ta_1,\ ta_2,\ ta_3, \qquad t^{-1}b_1,\ t^{-1}b_2,\ t^{-1}b_3.$$

Equivalently, in homogeneous coordinates $[X : Z]$ on $\mathbb{P}^1$, consider

$$C_t : \qquad Y^2 = \prod_{i=1}^{3}(X - ta_iZ)\prod_{j=1}^{3}(tX - b_jZ).$$

(a) Show that for $t \neq 0$, C_t is a smooth curve of genus 2.

(b) Show that as $t \to 0$ the branch divisor specializes to

$$3[0] + 3[\infty].$$

(c) Show that the corresponding naive limit is the bicuspidal curve

$$C_0 : \qquad Y^2 = X^3 Z^3.$$

Check that its two singularities are cusps, locally analytically of the form

$$y^2 = x^3.$$

(d) Show that the normalization of C_0 is $\mathbb{P}^1$. Using $\delta(y^2 = x^3) = 1$, verify that

$$p_a(C_0) = 2.$$

(e) Show that C_0 is not stable. In particular, show that it has a positive-dimensional automorphism group.

(f) By stable reduction, after a finite base change the same generic family also admits a stable limit C_0^{st}. Explain why C_0^{st} cannot be isomorphic to the bicuspidal curve C_0, and conclude that if one enlarges the moduli problem of genus-2 curves by allowing both stable curves and bicuspidal curves as limits, then the resulting moduli problem fails the valuative criterion for separatedness.

(g) Find the stable limit C_0^{st} of this family.

Problem 6. Genus 3: plane quartics and hyperelliptic curves

Let C be a smooth projective curve of genus 3. Its canonical linear system defines

$$\phi_K : C \longrightarrow \mathbb{P}H^0(C, \omega_C)^\vee \simeq \mathbb{P}^2.$$

(a) Assume that C is not hyperelliptic. Show that ϕ_K is an embedding. Since

$$\deg K_C = 4,$$

deduce that the canonical image is a smooth plane quartic.

(b) Conversely, let $C \subset \mathbb{P}^2$ be a smooth plane quartic. Use adjunction to show that

$$\omega_C \simeq \mathcal{O}_C(1),$$

so that its plane embedding is the canonical embedding.

(c) Now assume that C is hyperelliptic, with degree-2 map

$$h : C \longrightarrow \mathbb{P}^1.$$

Show that

$$\omega_C \simeq h^*\mathcal{O}_{\mathbb{P}^1}(2).$$

(d) Deduce that the canonical map factors as

$$C \xrightarrow{2:1} \mathbb{P}^1 \xrightarrow{|\mathcal{O}(2)|} Q \subset \mathbb{P}^2,$$

where Q is a smooth conic. Thus a hyperelliptic genus-3 curve is canonically a double cover of a conic rather than a plane quartic.

(e) Use Riemann–Hurwitz to show that the double cover is branched at 8 points on the conic.

(f) Compute the dimension of the plane-quartic locus by starting from

$$\mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}(4)) \simeq \mathbb{P}^{14}$$

and quotienting by PGL_3. Obtain

$$14 - 8 = 6 = 3g - 3.$$

(g) Compute the dimension of the hyperelliptic locus from 8 unordered points of $\mathbb{P}^1$ modulo PGL_2. Deduce that its dimension is 5 and hence that the hyperelliptic locus is a divisor in $\mathcal{M}_3$.

The main themes are: stable combinatorics, finite base change and normalization, resolution, failure of separatedness for the wrong compactification, and concrete projective models of curves.