HODGE THEORY AND MODULI SPACES

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Contents

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1. Lecture 1: Elliptic curves, periods, and $\mathcal{A}_1$

1.1. Riemann surfaces and elliptic curves.

Definition 1.1. A *Riemann surface* is a connected Hausdorff space X admitting an open cover $\{U_i\}_{i\in I}$ and homeomorphisms

$$\phi_i : U_i \to \phi_i(U_i) \subset \mathbb{C}$$

onto open subsets, such that every transition map

$$t_{ij} := \phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$$

is holomorphic.

Write $z_i \in \mathbb{C}$ for the coordinate on U_i supplied by the chart ϕ_i. On the double overlap $U_i \cap U_j$ we have $z_j = t_{ij}(z_i)$.

Definition 1.2. A *holomorphic* 1*-form* on X is a collection of local expressions $\{f_i(z_i)\, dz_i\}_{i\in I}$, with each f_i holomorphic, such that

$$f_i(z_i)\, dz_i = f_j(z_j)\, dz_j \qquad \text{on } U_i \cap U_j. \tag{1.1.1}$$

Since $z_j = t_{ij}(z_i)$, equality (1.1.1) is interpreted using

$$dz_j = d(t_{ij}(z_i)) = t'_{ij}(z_i)\, dz_i. \tag{1.1.2}$$

Thus the coefficient functions obey $f_i(z_i) = f_j(t_{ij}(z_i))t'_{ij}(z_i)$. This transformation rule allows us to view differential forms as sections of a holomorphic line bundle on X:

Definition 1.3. A *holomorphic line bundle* on X is a map $\pi : L \to X$ for which each restriction $\pi^{-1}(U_i) \to U_i$ is isomorphic to the product $U_i \times \mathbb{C} \to U_i$ for some open cover $\{U_i\}_{i\in I}$ of X and for which the changes of trivialization have the form

$$(x, v) \mapsto (x, g_{ij}(x)v),$$

where $g_{ij} : U_i \cap U_j \to \mathbb{C}^\times$ is a holomorphic non-vanishing function. Note that $g_{ii} = 1$ and $g_{ij}g_{jk} = g_{ik}$ on every triple overlap.

A holomorphic section of L is a holomorphically varying choice of a vector in each fiber; i.e. a holomorphic map $s\colon X \to L$ for which $\pi \circ s = \mathrm{id}_X$. The $\mathbb{C}$-vector space of such sections is denoted $H^0(X, L)$. In local trivializations, a section is given by holomorphic functions $s_i\colon U_i \to \mathbb{C}$ satisfying $s_j = g_{ij}s_i$ on $U_i \cap U_j$. The dual line bundle is denoted L^{-1} and it has transition functions g_{ij}^{-1}. The trivial line bundle $X \times \mathbb{C}$ is denoted $\mathcal{O}_X$.

Definition 1.4. The *canonical bundle* $K_X = \Omega^1_X$ is the holomorphic line bundle whose local frame on U_i is dz_i. The change-of-frame is $dz_j = t'_{ij}(z_i)dz_i$. Its holomorphic sections $H^0(X, K_X)$ are the holomorphic 1-forms on X.

Every holomorphic line bundle on a compact Riemann surface has a nonzero meromorphic section.

Definition 1.5. Let L be a holomorphic line bundle on a compact Riemann surface, and let s be a nonzero meromorphic section. The *degree* of L is

$$\deg L = \sum_{p \in X} \mathrm{ord}_p(s).$$

That is, the number of zeros of s minus the number of poles, counted with multiplicity. This integer is independent of the choice of s.

The genus of a compact Riemann surface X has a genus $g(X)$, the genus of its underlying oriented surface.

Theorem 1.6 (Riemann–Roch)**.** *Let L be a holomorphic line bundle on a compact Riemann surface X of genus g. Then*

$$\dim H^0(X, L) - \dim H^0(X, K_X \otimes L^{-1}) = \deg L + 1 - g. \tag{1.1.3}$$

In particular,

$$\dim H^0(X, K_X) = g, \qquad \deg K_X = 2g - 2.$$

Proof of the last two assertions. The two displayed consequences follow from the formula, first with $L = \mathcal{O}_X$ and then with $L = K_X$. □

For background on differential forms, Riemann–Roch, and Abel's theorem on a compact Riemann surface, see [For81, Sections 9, 16, and 20].

Definition 1.7. An *elliptic curve over* $\mathbb{C}$ is a compact Riemann surface E of genus one, together with a marked point $0 \in E$.

The marked point will be the identity for a group law.

Definition 1.8. A *lattice* in $\mathbb{C}$ is a discrete subgroup

$$\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2 \subset \mathbb{C}. \tag{1.1.4}$$

We may always choose generators of Λ for which $\mathrm{Im}(\omega_2/\omega_1) > 0$, and we call such a basis of Λ *oriented.*

The quotient $E_\Lambda = \mathbb{C}/\Lambda$ is a Riemann surface, admitting a holomorphic addition map, $E_\Lambda \times E_\Lambda \to E_\Lambda$ with addition induced from $\mathbb{C}$. A fundamental parallelogram is compact, and hence so is E_Λ. Topologically it is a 2-dimensional torus $E_\Lambda \simeq_{\mathrm{homeo}} S^1 \times S^1$.

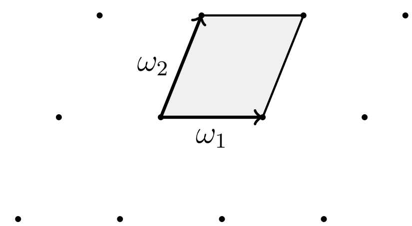

FIGURE 1. The lattice points are the integral linear combinations of the two displayed vectors.

Proposition 1.9. *If E is an elliptic curve, then $H^0(E,\Omega^1_E)$ is one-dimensional. Every nonzero $\omega \in H^0(E,\Omega^1_E)$ is nowhere vanishing.*

Proof. By Theorem 1.6, $\dim H^0(E,K_E) = 1$ and $\deg K_E = 0$. The zeros of a nonzero holomorphic 1-form have total multiplicity $\deg K_E = 0$, so there are no zeros. ☐

Fix a nonzero holomorphic 1-form $0 \neq \omega \in H^0(E,\Omega^1_E)$. Integration defines the period homomorphism

$$\begin{aligned} \mathrm{per}_\omega : H_1(E,\mathbb{Z}) &\to \mathbb{C}, \\ [\gamma] &\mapsto \int_\gamma \omega. \end{aligned} \tag{1.1.5}$$

Since the underlying surface is a torus, $H_1(E,\mathbb{Z}) \simeq \mathbb{Z}^2$.

Lemma 1.10 (The genus-one bilinear relation)**.** *Let α,β be an oriented basis of $H_1(E,\mathbb{Z})$, and set*

$$\omega_1 = \int_\alpha \omega, \qquad \omega_2 = \int_\beta \omega.$$

Then

$$\mathrm{Im}(\overline{A}B) = \frac{i}{2}\int_E \omega\wedge\overline{\omega} > 0. \tag{1.1.6}$$

Proof. The equality follows by cutting E along α and β, choosing a primitive $\omega = dF$ on the resulting quadrilateral, and applying Stokes' theorem. The form $\frac{i}{2}\omega\wedge\overline{\omega}$ is an everywhere positive area form because ω never vanishes. ☐

Theorem 1.11 (Uniformization in genus one)**.** *The image Λ_ω of (1.1.5) is a lattice, and integration gives an isomorphism of pointed Riemann surfaces*

$$\begin{aligned} \mathrm{AJ}_\omega : E &\xrightarrow{\sim} \mathbb{C}/\Lambda_\omega \\ p &\mapsto \textstyle\int_0^p \omega \pmod{\Lambda_\omega}. \end{aligned} \tag{1.1.7}$$

The pullback of dz is ω.

Proof. Changing the path from 0 to p changes the integral by a period, so the map is well-defined. If α, β is an oriented basis of $H_1(E, \mathbb{Z})$ and $\omega_1 = \int_\alpha \omega$, $\omega_2 = \int_\beta \omega$, then Lemma 1.10 shows that ω_1 and ω_2 are linearly independent over $\mathbb{R}$. Their integral span is therefore a lattice. The map (1.1.7) has derivative ω, so it is a local biholomorphism. It induces an isomorphism on the singular homology group H_1 and hence is a degree-one covering. It is therefore an isomorphism. ☐

The complex uniformization of elliptic curves and its relation with Weierstrass equations are developed further in [Sil09, Chapter VI].

Multiplying ω by $c \in \mathbb{C}^\times$ scales $\Lambda_\omega \mapsto c\Lambda_\omega = \Lambda_{c\omega}$. Thus an elliptic curve uniquely determines a *homothety class* of lattices $\Lambda \subset \mathbb{C}$ under the equivalence relation $\Lambda \sim c\Lambda$, for $c \in \mathbb{C}^\times$.

1.2. **The Hodge decomposition and the period coordinate.** If X is compact, every holomorphic 1-form is closed and hence determines a class in $H^1(X, \mathbb{C})$ by de Rham's theorem. This map is injective: if $\omega = dF$, then Stokes' theorem gives

$$\frac{i}{2}\int_X \omega \wedge \overline{\omega} = \frac{i}{2}\int_X d(F\, d\overline{F}) = 0,$$

which forces $\omega = 0$ as $\frac{i}{2}\omega \wedge \bar{\omega}$ is pointwise non-negative.

Definition 1.12. For a compact Riemann surface X, define

$$H^{1,0}(X) \coloneqq \{[\omega] \in H^1(X, \mathbb{C}) : \omega \text{ is a holomorphic 1-form}\},$$
$$H^{0,1}(X) \coloneqq \{[\overline{\omega}] \in H^1(X, \mathbb{C}) : \omega \text{ is a holomorphic 1-form}\}.$$

Thus complex conjugation interchanges $H^{1,0}(X)$ and $H^{0,1}(X)$.

Define the upper half-plane $\mathfrak{H}$ and, for each $\tau \in \mathfrak{H}$, the corresponding complex torus by

$$\mathfrak{H} \coloneqq \{\tau \in \mathbb{C} : \operatorname{Im}\tau > 0\}, \qquad E_\tau \coloneqq \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau). \tag{1.2.1}$$

Let α be the image of the segment $[0, 1]$ and β the image of $[0, \tau]$ in E_τ. These form 1-cycles, giving an oriented basis of $H_1(E_\tau, \mathbb{Z})$. Let α^*, β^* be the dual integral cohomology basis, of $H^1(E_\tau, \mathbb{Z})$.

Proposition 1.13 (The Hodge decomposition for an elliptic curve)**.** *There is a direct sum decomposition*

$$H^1(E_\tau, \mathbb{C}) = H^{1,0}(E_\tau) \oplus H^{0,1}(E_\tau), \tag{1.2.2}$$

where

$$\begin{aligned} H^{1,0}(E_\tau) &= \mathbb{C}(\alpha^* + \tau\beta^*), \\ H^{0,1}(E_\tau) &= \mathbb{C}(\alpha^* + \overline{\tau}\beta^*). \end{aligned} \tag{1.2.3}$$

Moreover,

$$\frac{i}{2}\int_{E_\tau} dz \wedge d\bar{z} = \operatorname{Im}\tau > 0. \tag{1.2.4}$$

Proof. By Proposition 1.9, dz spans the holomorphic 1-forms on E_τ. Its periods along α, β are $1, \tau$, so

$$[dz] = \alpha^* + \tau\beta^*,$$
$$[d\bar{z}] = \alpha^* + \bar{\tau}\beta^*.$$

The two classes are independent because $\tau \neq \bar{\tau}$. Finally, (1.2.4) is Lemma 1.10 with $\omega_1 = 1$ and $\omega_2 = \tau$. $\square$

Definition 1.14. Suppose that an elliptic curve is equipped with an oriented basis α, β of its first homology. Normalize a nonzero holomorphic form ω by $\omega_1 = \int_\alpha \omega = 1$. Its *period coordinate* is

$$\tau = \frac{\omega_2}{\omega_1} = \frac{\int_\beta \omega}{\int_\alpha \omega} = \int_\beta \omega \in \mathfrak{H}.$$

Equivalently, τ is the slope of the line $H^{1,0}(E)$ in the integral basis α^*, β^* of $H^1(E, \mathbb{C})$.

The upper half-plane $\mathfrak{H}$ is the first instance of a *period domain*: a space parameterizing Hodge structures subject to a positivity condition; see Lecture 2. After choosing an oriented integral basis of $H^1(E, \mathbb{Z})$, the positivity condition cuts out $\mathfrak{H}$ from $\mathbb{P}(H^1(E, \mathbb{C})) \simeq \mathbb{P}^1$.

The realization of a 1-dimensional complex torus as a plane cubic, and the description of its group law by divisors, are developed in Exercises 1.19 and 1.20.

1.3. **Lattices and the modular quotient.**

Proposition 1.15. *Two complex tori $\mathbb{C}/\Lambda$ and $\mathbb{C}/\Lambda'$ are isomorphic if and only if $\Lambda' = a\Lambda$ for some $a \in \mathbb{C}^\times$.*

Proof. See Exercise 1.21, part (1). $\square$

Scaling an oriented lattice basis so its first vector is 1, we see that every homothety class is represented by $\Lambda_\tau = \mathbb{Z} \oplus \mathbb{Z}\tau$ for some $\tau \in \mathfrak{H}$. Changing the oriented integral basis changes τ by

$$\gamma \cdot \tau = \frac{a\tau + b}{c\tau + d}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}). \tag{1.3.1}$$

It follows that the set of isomorphism classes of complex elliptic curves is in bijection with

$$\mathcal{A}_1(\mathbb{C}) = \mathrm{SL}_2(\mathbb{Z}) \backslash \mathfrak{H}. \tag{1.3.2}$$

The standard fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathfrak{H}$ by fractional linear transformations (1.3.1) is

$$\mathcal{F} = \{\tau \in \mathfrak{H} : |\operatorname{Re}\tau| \leq \tfrac{1}{2},\ |\tau| \geq 1\}. \tag{1.3.3}$$

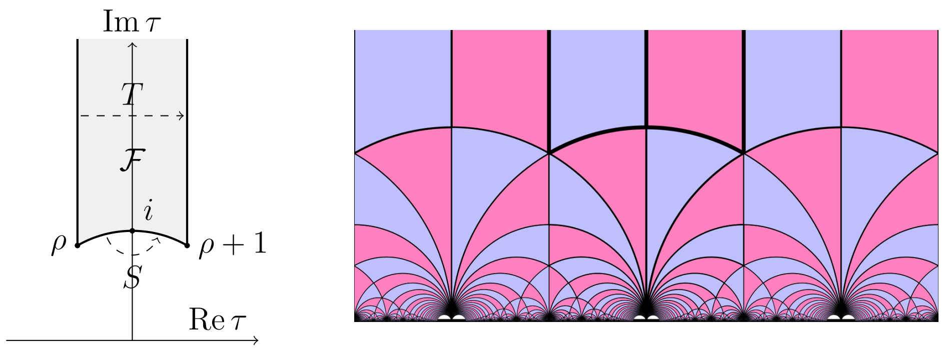

FIGURE 2. Left: The standard fundamental domain. The vertical sides are identified by $T \cdot \tau = \tau + 1$, and the two circular arcs are identified by $S \cdot \tau = -1/\tau$. Right: the $\mathrm{PSL}_2(\mathbb{Z})$-images of $\mathcal{F}$ tiling the upper half plane; figure by [Neo22].

The element $-I$ acts trivially on $\mathfrak{H}$, so the action factors through

$$\mathrm{PSL}_2(\mathbb{Z}) \coloneqq \mathrm{SL}_2(\mathbb{Z})/\{\pm I\}.$$

The stabilizer of i in $\mathrm{PSL}_2(\mathbb{Z})$ has order 2, while that of $\rho = e^{2\pi i/3}$ has order 3. There are local coordinates u_i at i and u_ρ at ρ in which the stabilizers act by $u_i \mapsto -u_i$ and $u_\rho \mapsto \rho^2 u_\rho$. Thus u_i^2 and u_ρ^3 are local coordinates on the ordinary quotient. These coordinates are constructed in Exercise 1.22, part (3).

Over $\mathfrak{H}$, there is a family of elliptic curves

$$\mathcal{E}_{\mathfrak{H}} = (\mathfrak{H} \times \mathbb{C})/\mathbb{Z}^2, \qquad (m,n)\cdot(\tau,z) = (\tau, z + m + n\tau). \tag{1.3.4}$$

The group $\mathrm{SL}_2(\mathbb{Z})$ acts on this family by

$$\gamma \cdot (\tau, z) = \left(\gamma\cdot\tau, \frac{z}{c\tau + d}\right). \tag{1.3.5}$$

Warning 1.16. The ordinary quotient

$$(\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2)\backslash(\mathfrak{H}\times\mathbb{C}) \to \mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}$$

is not a universal family of elliptic curves over the ordinary quotient space. See Exercise 1.21, part (2), for what goes wrong in the fibers.

Observe that the quotient in (1.3.2) is noncompact. The group $\mathrm{SL}_2(\mathbb{Z})$ acts transitively on $\mathbb{P}^1(\mathbb{Q})$, so adjoining the rational boundary points adds a single point:

$$X(1) = \mathrm{SL}_2(\mathbb{Z})\backslash\big(\mathfrak{H} \cup \mathbb{P}^1(\mathbb{Q})\big). \tag{1.3.6}$$

This added point is called the *cusp*. We now describe its neighborhoods and thereby give $X(1)$ its Riemann surface structure at the cusp.

Proposition 1.17 (A cusp neighborhood)**.** *Fix $C > 1$, and set*

$$\mathfrak{H}_C = \{\tau \in \mathfrak{H} : \operatorname{Im}\tau > C\}.$$

Let U_C be the image of $\mathfrak{H}_C$ in $\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}$. The only identifications among points of $\mathfrak{H}_C$ are induced by powers of $T \cdot \tau = \tau + 1$. Moreover, $q = e^{2\pi i\tau}$ induces an isomorphism

$$\langle T\rangle\backslash\mathfrak{H}_C \xrightarrow{\sim} \{q \in \mathbb{C} : 0 < |q| < e^{-2\pi C}\}.$$

Thus $U_C \cup \{cusp\}$ is a disk, with $q = 0$ at the cusp. As C varies, these disks form a neighborhood basis of the cusp in $X(1)$.

Proof. Suppose that $\tau, \gamma\cdot\tau \in \mathfrak{H}_C$, where

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

If $c \neq 0$, then

$$\operatorname{Im}(\gamma\cdot\tau) = \frac{\operatorname{Im}\tau}{|c\tau + d|^2} \leq \frac{1}{c^2 \operatorname{Im}\tau} < \frac{1}{C} < C,$$

a contradiction. Hence $c = 0$, which shows that γ acts as a power of T. The function q is invariant under T and q exactly identifies T-orbits. We have $|q| = e^{-2\pi \operatorname{Im}\tau}$ and so q is valued in the stated punctured disk; $q = 0$ fills in the puncture. □

The algebraic coordinate j and a nodal curve parameterized by the cusp are described in Exercises 1.23 and 1.24.

More generally we can consider the so-called "congruence subgroups" and their associated quotients:

Definition 1.18. For $N \geq 1$, define the congruence subgroups

$$\Gamma(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a, d \equiv 1 \text{ and } b, c \equiv 0 \pmod{N}\right\},$$

$$\Gamma_1(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a, d \equiv 1 \text{ and } c \equiv 0 \pmod{N}\right\},$$

$$\Gamma_0(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N}\right\}.$$

For $\Gamma = \Gamma(N), \Gamma_1(N), \Gamma_0(N)$, put

$$Y_\Gamma = \Gamma\backslash\mathfrak{H}, \qquad X_\Gamma = \Gamma\backslash\big(\mathfrak{H} \cup \mathbb{P}^1(\mathbb{Q})\big),$$

and write these compact Riemann surfaces as $X(N)$, $X_1(N)$, $X_0(N)$, for $\Gamma = \Gamma(N)$, $\Gamma_1(N)$, $\Gamma_0(N)$ respectively.

Away from the cusps, $X(N)$ may be interpreted as an elliptic curve equipped with an oriented basis of its N-torsion, $X_1(N)$ as an elliptic curve equipped with a point of order exactly N, and $X_0(N)$ as an elliptic curve with a cyclic subgroup of order N.

For modular curves and these moduli interpretations, see [DS05, Chapters 1–3].

1.4. **Problem Set.**

Exercise 1.19 (Elliptic functions and the Weierstrass cubic)**.** Let $\Lambda \subset \mathbb{C}$ be a lattice and $E = \mathbb{C}/\Lambda$. An *elliptic function* for Λ is a meromorphic function on $\mathbb{C}$ invariant under translation by Λ.

(1) Prove that an entire elliptic function is constant.

(2) Define the Weierstrass function by

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{0\neq\lambda\in\Lambda}\left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right). \tag{1.4.1}$$

Prove that the series converges uniformly on compact subsets of $\mathbb{C}\setminus\Lambda$ and that $\wp_\Lambda$ is elliptic and even. Determine the poles of $\wp_\Lambda$ and $\wp'_\Lambda$.

(3) Set

$$\begin{aligned} g_2(\Lambda) &= 60\textstyle\sum_{0\neq\lambda\in\Lambda}\lambda^{-4},\\ g_3(\Lambda) &= 140\textstyle\sum_{0\neq\lambda\in\Lambda}\lambda^{-6}. \end{aligned} \tag{1.4.2}$$

Compute the Laurent expansions of $\wp$ and $\wp'$ at 0, and use part (1) to prove

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3. \tag{1.4.3}$$

(4) Prove that $\wp(z) = \wp(w)$ if and only if $w = \pm z$ in E, and that equality of both $\wp$ and $\wp'$ forces $z = w$. Deduce that

$$\begin{aligned} \Phi_\Lambda : E &\to \mathbb{P}^2,\\ z &\mapsto [\wp(z) : \wp'(z) : 1] \end{aligned} \tag{1.4.4}$$

extends over 0 and identifies E with the smooth cubic

$$Y^2Z = 4X^3 - g_2XZ^2 - g_3Z^3. \tag{1.4.5}$$

Show that 0 maps to $[0:1:0]$ and that

$$\Delta(\Lambda) = g_2(\Lambda)^3 - 27g_3(\Lambda)^2 \tag{1.4.6}$$

is nonzero.

(5) Write $\mathcal{O}_E(n[0])$ for the line bundle whose local holomorphic sections are meromorphic functions having no poles away from 0 and a pole of order at most n at 0. Use Riemann–Roch to determine $H^0(E, \mathcal{O}_E(n[0]))$ for $n = 1, 2, 3$, and recover the map (1.4.4) from the complete linear system $|3[0]|$.

Exercise 1.20 (Divisors of elliptic functions)**.** A *divisor* on $E = \mathbb{C}/\Lambda$ is a finite sum $D = \sum_p n_p[p]$ with $n_p \in \mathbb{Z}$; its degree is $\sum_p n_p$. The *principal divisor* of a nonzero meromorphic function is

$$\operatorname{div}(f) = \sum_p \operatorname{ord}_p(f)[p].$$

Two divisors are *linearly equivalent* if their difference is principal, and $\operatorname{Pic}^0(E)$ is the group of degree zero divisors modulo linear equivalence. Let P be a fundamental parallelogram whose boundary contains no zero or pole of the elliptic function f.

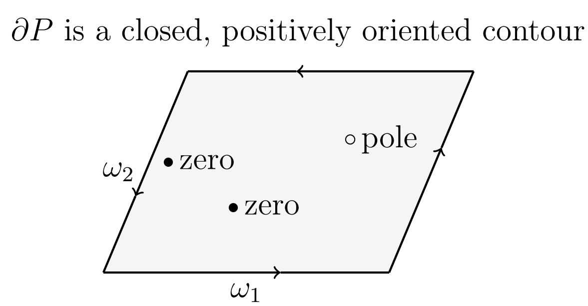

FIGURE 3. The contour used to count the zeros and poles of an elliptic function.

(1) Use the residue theorem on the contour in Figure 3 to prove

$$\sum_p \operatorname{ord}_p(f) = 0, \qquad \sum_p \operatorname{ord}_p(f)p = 0 \in E. \tag{1.4.7}$$

(2) Let $a \in \mathbb{C}$ and choose $z_a \in E$ with $\wp(z_a) = a$. Prove

$$\operatorname{div}(\wp - a) = [z_a] + [-z_a] - 2[0],$$
$$\operatorname{div}(\wp') = \textstyle\sum_{0 \neq z \in E[2]}[z] - 3[0].$$

(3) Use either Riemann–Roch or (2) to prove the converse to (1): if $D = \sum_p n_p[p]$ satisfies

$$\sum_p n_p = 0, \qquad \sum_p n_p p = 0 \in E,$$

then D is principal. Deduce that

$$\begin{aligned} \mathrm{Pic}^0(E) &\to E, \\ \sum_p n_p[p] &\mapsto \sum_p n_p p \end{aligned}$$

is an isomorphism.

(4) Let $C \subset \mathbb{P}^2$ be the cubic in (1.4.5), with $0 = [0 : 1 : 0]$, and use (1.4.4) to identify C with E. Let $\ell = \{L(X, Y, Z) = 0\} \subset \mathbb{P}^2$ be a line, and write

$$\ell \cap C = \{p, q, r\},$$

where repeated points record intersection multiplicity. Prove that the meromorphic function L/Z on C has

$$\mathrm{div}(L/Z) = [p] + [q] + [r] - 3[0].$$

Deduce from part (3) that $p + q + r = 0$ in E.

Exercise 1.21 (Homotheties)**.**

(1) Prove Proposition 1.15 by lifting a pointed isomorphism to the universal covers $\mathbb{C} \to \mathbb{C}/\Lambda$ and $\mathbb{C} \to \mathbb{C}/\Lambda'$.

(2) What is the fiber over $[\tau] \in \mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}$ of the map in Warning 1.16? Is it isomorphic to the elliptic curve E_τ?

Exercise 1.22 (Local quotient coordinates)**.**

(1) Prove that $\mathcal{F}$ in (1.3.3), with the side pairings described there, is a fundamental domain for the action of $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathfrak{H}$.

(2) Determine the stabilizers of i and $\rho = e^{2\pi i/3}$ in $\mathrm{PSL}_2(\mathbb{Z})$.

(3) Find local coordinates u_i at i and u_ρ at ρ in which the respective stabilizers act by $u_i \mapsto -u_i$ and $u_\rho \mapsto \rho^2 u_\rho$. Deduce local coordinates on the ordinary quotient near $[i]$ and $[\rho]$.

Exercise 1.23 (Modular forms and the j-line)**.** This exercise follows the concise contour-theoretic treatment in [Ser73, Chapter VII, Sections 1–4]. A *modular form of weight* k for $\mathrm{SL}_2(\mathbb{Z})$ is a holomorphic function $f : \mathfrak{H} \to \mathbb{C}$ such that

$$f(\gamma \cdot \tau) = (c\tau + d)^k f(\tau), \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), \tag{1.4.8}$$

and whose Fourier expansion $\sum_{n\geq 0} c_n q^n$ in $q = e^{2\pi i \tau}$ has no negative powers. We call f a *cusp form* if the constant coefficient $c_0 = 0$ vanishes. Denote the spaces of modular forms and cusp forms by M_k and S_k, respectively. For $2r \geq 4$, set

$$G_{2r}(\tau) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} (m + n\tau)^{-2r}. \tag{1.4.9}$$

Normalize

$$\begin{aligned} E_4 &:= (2\zeta(4))^{-1} G_4 = 1 + 240q + O(q^2), \\ E_6 &:= (2\zeta(6))^{-1} G_6 = 1 - 504q + O(q^2), \end{aligned} \tag{1.4.10}$$

and set

$$\Delta = \frac{E_4^3 - E_6^2}{1728} = q + O(q^2). \tag{1.4.11}$$

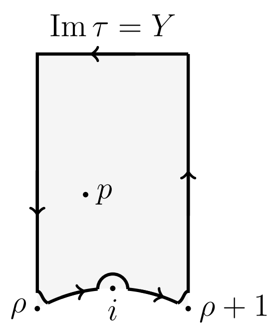

FIGURE 4. A closed, positively oriented contour following the boundary of a truncated fundamental domain, with small inward detours around the elliptic points.

(1) Prove that $G_{2r} \in M_{2r}$ for $2r \geq 4$, and determine its constant Fourier coefficient.
(2) For $0 \neq f \in M_k$ prove the valence formula

$$\mathrm{ord}_\infty(f) + \frac{1}{2}\,\mathrm{ord}_i(f) + \frac{1}{3}\,\mathrm{ord}_\rho(f) + \sum_{\substack{p\in\mathrm{SL}_2(\mathbb{Z})\setminus\mathfrak{H} \\ p\neq i,\rho}} \mathrm{ord}_p(f) = \frac{k}{12} \tag{1.4.12}$$

by integrating $f'(\tau)/f(\tau)\,d\tau$ over the contour in Figure 4.

(3) Deduce that E_4 has a simple zero at $\mathrm{SL}_2(\mathbb{Z})\cdot\rho$ and no other zero, that E_6 has a simple zero at $\mathrm{SL}_2(\mathbb{Z})\cdot i$ and no other zero, and that Δ has a simple zero at the cusp and no zero on $\mathfrak{H}$.

(4) Prove

$$M_\bullet = \bigoplus_{k\geq 0} M_k = \mathbb{C}[E_4, E_6]. \tag{1.4.13}$$

As a strategy, first use the divisor of Δ to identify cusp forms of weight k with ΔM_{k-12}. Then subtract from a modular form a suitable monomial in E_4 and E_6 having the same constant term, and induct on the weight. Finally, prove that E_4 and E_6 are algebraically independent.

(5) Define

$$j(\tau) = \frac{E_4(\tau)^3}{\Delta(\tau)}. \tag{1.4.14}$$

Prove that j gives an isomorphism

$$j : X(1) \xrightarrow{\sim} \mathbb{P}^1$$

which takes the cusp to ∞. Determine $j(\rho)$ and $j(i)$, compute the first two terms of the Fourier expansion of j, and use the Weierstrass form (1.4.5) to describe the additional automorphisms of the hexagonal and square lattices in algebraic terms.

Exercise 1.24 (A nodal cubic at the cusp)**.** For $|t|$ small, choose the holomorphic square root $b(t) = \sqrt{1-1728t}$ with $b(0) = 1$, and let $\mathcal{E}_t$ be the projective closure of

$$y^2 = 4x^3 - 3x - b(t), \tag{1.4.15}$$

obtained by adding $[0:1:0]$.

(1) Determine the cubic discriminant and thus the singular fibers of (1.4.15). Prove that the central fiber has a node.

(2) For a smooth cubic $y^2 = 4x^3 - g_2 x - g_3$, set

$$j = 1728\frac{g_2^3}{g_2^3 - 27g_3^2}. \tag{1.4.16}$$

Compute $j(\mathcal{E}_t)$. Using the j-line description proved in Exercise 1.23, show that $t \mapsto [\mathcal{E}_t]$ identifies a small disk centered at $t = 0$ with a neighborhood of the cusp in $X(1)$.

(3) Using the Fourier expansion of j obtained in Exercise 1.23, show that the analytic coordinate q and the algebraic coordinate $t = 1/j$ satisfy
$$t = q + O(q^2).$$

2. Lecture 2: Hodge structures

2.1. Integral Hodge structures.

The decomposition of $H^1(E, \mathbb{C})$ in Proposition 1.13 is not special to elliptic curves. We begin by isolating its linear-algebraic features.

Definition 2.1. An *integral Hodge structure of weight k* is a finitely generated abelian group $H_{\mathbb{Z}}$ together with a decomposition

$$H_{\mathbb{C}} := H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{p+q=k} H^{p,q} \qquad \text{such that} \qquad \overline{H^{p,q}} = H^{q,p}. \tag{2.1.1}$$

The integers $h^{p,q} = \dim_{\mathbb{C}} H^{p,q}$ are its *Hodge numbers*.

The same information can be arranged into a decreasing filtration.

Definition 2.2. The *Hodge filtration* of (2.1.1) is

$$F^p H_{\mathbb{C}} = \bigoplus_{r \geq p} H^{r,k-r}. \tag{2.1.2}$$

The decomposition may be recovered from the filtration by

$$H^{p,q} = F^p H_{\mathbb{C}} \cap \overline{F^q H_{\mathbb{C}}}. \tag{2.1.3}$$

We will use filtrations in Lecture 3, since the subspaces F^p vary holomorphically in a family of smooth projective varieties, even though the summands $H^{p,q}$ generally do not.

2.2. Forms, Dolbeault cohomology, and Hodge decomposition.

Let X be a complex manifold of dimension n (i.e. Definition 1.1, but where the charts are valued in $\mathbb{C}^n$ instead of $\mathbb{C}$). In local coordinates $(z_1, \ldots, z_n)$ on a chart U, every smooth complex-valued k-form has a unique decomposition into forms of type (p, q) with $p + q = k$.

Definition 2.3. A smooth *(p, q)-form* on X is a smooth differential form which is locally of the shape

$$\alpha = \sum_{\substack{|I|=p \\ |J|=q}} f_{I,J}(z, \overline{z})\, dz_I \wedge d\overline{z}_J. \tag{2.2.1}$$

We denote the vector space of such forms by $\mathcal{A}^{p,q}(X)$.

Formula 1.1.2 generalizes to n variables as a way to relate the differentials dz_i in different charts (via the Jacobian matrix of the transition function). That is, holomorphic changes of coordinates do not mix the differentials dz_i with the differentials $d\overline{z}_j$. So the type of a form is independent of the chosen coordinates. The exterior derivative d in de

Rham cohomology therefore splits as

$$d = \partial + \overline{\partial}, \tag{2.2.2}$$

$$\partial : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p+1,q}(X), \qquad \overline{\partial} : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q+1}(X). \tag{2.2.3}$$

The identity $d^2 = 0$ implies $\partial^2 = \overline{\partial}^2 = 0$ and $\partial\overline{\partial} + \overline{\partial}\partial = 0$.

Definition 2.4. The *holomorphic cotangent bundle* Ω^1_X is the holomorphic vector bundle whose local sections are the forms

$$\sum_{j=1}^{n} f_j(z)\, dz_j \tag{2.2.4}$$

with each f_j holomorphic. We set

$$\Omega^p_X = \bigwedge^p \Omega^1_X. \tag{2.2.5}$$

Its local holomorphic sections are the holomorphic p-forms

$$\sum_{|I|=p} f_I(z)\, dz_I. \tag{2.2.6}$$

In particular, $H^0(X, \Omega^p_X)$ is the space of global holomorphic p-forms. It coincides with the kernel of $\overline{\partial}\colon \mathcal{A}^{p,0}(X) \to \mathcal{A}^{p,1}(X)$.

Definition 2.5. The *Dolbeault cohomology* of X is

$$H^{p,q}_{\overline{\partial}}(X) = \frac{\ker\big(\overline{\partial} : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q+1}(X)\big)}{\operatorname{im}\big(\overline{\partial} : \mathcal{A}^{p,q-1}(X) \to \mathcal{A}^{p,q}(X)\big)}. \tag{2.2.7}$$

In particular, we have $H^{p,0}_{\overline{\partial}}(X) = H^0(X, \Omega^p_X)$. The sheaf higher cohomology groups of Ω^p_X also have a description in terms of (p,q)-forms as follows:

Theorem 2.6 (Dolbeault theorem)**.** *For every complex manifold X, there is a natural isomorphism*

$$H^q(X, \Omega^p_X) \xrightarrow{\sim} H^{p,q}_{\overline{\partial}}(X). \tag{2.2.8}$$

The theorem follows from the *Poincaré $\overline{\partial}$-lemma*, which translates into the fact that the Dolbeault complex $(\mathcal{A}^{p,\bullet}(X), \overline{\partial})$ is an acyclic resolution of the sheaf Ω^p.

Theorem 2.7 (Hodge decomposition theorem)**.** *Let X be a smooth projective complex variety. Then*

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X), \qquad H^{p,q}(X) \simeq H^q(X, \Omega^p_X). \tag{2.2.9}$$

Complex conjugation exchanges $H^{p,q}(X)$ and $H^{q,p}(X)$. Consequently, $H^k(X,\mathbb{Z})$ carries a natural integral Hodge structure of weight k.

Proof Ideas. By definition, classes in $H^{p,q}_{\overline{\partial}}(X)$ are represented by $\overline{\partial}$-closed (p,q)-forms. A key part of the proof is to choose auxiliary data on X called a *Kähler metric*. First, this choice allows us to define an adjoint $\overline{\partial}^\dagger$ to the operator $\overline{\partial}$. Analysis is used prove that every element $a \in H^{p,q}_{\overline{\partial}}(X)$ lifts uniquely to a (p,q)-form $\alpha \in \mathcal{A}^{p,q}(X)$ satisfying

$$\overline{\partial}\alpha = \overline{\partial}^\dagger\alpha = 0.$$

Via the so-called *Kähler identities*, one deduces that α lies also in the kernel of ∂, and hence $d\alpha = 0$. This lift therefore defines a map

$$\begin{aligned} H^{p,q}_{\overline{\partial}}(X) &\to H^k(X,\mathbb{C}), \\ a &\mapsto [\alpha]. \end{aligned}$$

The Kähler identities are again used to prove this map is injective for each (p,q), the images are complementary, and they form a direct sum decomposition of $H^k(X,\mathbb{C})$. $\square$

It is worth noting that the decomposition (2.2.9) is independent of the chosen Kähler metric.

Example 2.8. If $\eta \in H^2(\mathbb{P}^n,\mathbb{Z})$ is the class of a hyperplane, then

$$H^\bullet(\mathbb{P}^n,\mathbb{Z}) = \mathbb{Z}[\eta]/(\eta^{n+1}), \qquad \eta \in H^{1,1}(\mathbb{P}^n). \tag{2.2.10}$$

Thus $H^{p,q}(\mathbb{P}^n) = 0$ for $p \neq q$ and $H^{p,p}(\mathbb{P}^n) = \mathbb{C}\eta^p$.

Example 2.9 (A smooth cubic threefold)**.** Let $X \subset \mathbb{P}^4$ be a smooth hypersurface of degree 3. The Lefschetz hyperplane theorem and Poincaré duality determine the cohomology outside the middle degree. The adjunction formula gives

$$K_X \simeq \mathcal{O}_X(-2), \tag{2.2.11}$$

so $h^{3,0}(X) = 0$. The only remaining Hodge number is $h^{2,1}(X)$; in Exercise 2.17 we compute

$$h^{2,1}(X) = h^{1,2}(X) = 5. \tag{2.2.12}$$

The Hodge numbers of X can be arranged in a *Hodge diamond*, see Figure 5.

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 1 & & 0 & \\
0 & & 5 & & 5 & & 0 \\
 & 0 & & 1 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

FIGURE 5. Hodge diamond of a smooth cubic threefold

2.3. **Polarizations, hard Lefschetz and primitive cohomology.**

Definition 2.10. A holomorphic line bundle $\mathcal{L}$ on a projective variety X is *very ample* if its global sections define a closed embedding

$$[s_0 : \cdots : s_N] \colon X \to \mathbb{P}^N,$$

$(s_i)_{i=0,\ldots,N}$ a basis of $H^0(X, \mathcal{L})$. It is *ample* if $\mathcal{L}^{\otimes m}$ is very ample for some positive integer $m > 0$.

Let X be smooth projective of dimension n, let $\mathcal{L}$ be ample, and set

$$\eta = c_1(\mathcal{L}) \in H^2(X, \mathbb{Z}) \cap H^{1,1}(X), \qquad L(\alpha) = \eta \wedge \alpha. \tag{2.3.1}$$

Here $c_1(\mathcal{L})$ is the *Chern class*, which assigns to $\mathcal{L}$ the cohomology class of the closed $(1,1)$-form $\frac{i}{2\pi}\partial\overline{\partial}\log h(s, \overline{s})$ where $h \colon \mathcal{L} \otimes \overline{\mathcal{L}} \to \mathcal{C}^\infty$ is a hermitian metric and s is a local frame of $\mathcal{L}$ (exercise: this form is well-defined, independent of the choice of local frame s).

Theorem 2.11 (Hard Lefschetz theorem)**.** *For $0 \leq k \leq n$, cup product with η^{n-k} is an isomorphism*

$$L^{n-k} : H^k(X, \mathbb{Q}) \xrightarrow{\sim} H^{2n-k}(X, \mathbb{Q}). \tag{2.3.2}$$

In particular, the Hodge diamond, see Figure 5, is symmetric under flipping along the x-axis (in addition to the Hodge symmetry which gives symmetry under flipping along the y-axis). The part of the cohomology which is genuinely new in each degree is the following.

Definition 2.12. For $0 \leq k \leq n$, the *primitive cohomology* is

$$P^k(X, \mathbb{Q}) = \ker\bigl(L^{n-k+1} : H^k(X, \mathbb{Q}) \to H^{2n-k+2}(X, \mathbb{Q})\bigr). \tag{2.3.3}$$

Its integral lattice is

$$P^k(X, \mathbb{Z}) = P^k(X, \mathbb{Q}) \cap H^k(X, \mathbb{Z}). \tag{2.3.4}$$

Hard Lefschetz implies the *Lefschetz decomposition*

$$H^k(X,\mathbb{Q}) = \bigoplus_{r\geq 0} L^r P^{k-2r}(X,\mathbb{Q}) \qquad (0 \leq k \leq n). \tag{2.3.5}$$

Since η has type $(1,1)$, the operator L sends $H^{p,q} \to H^{p+1,q+1}$. It follows that $P^k(X,\mathbb{Z})$ defines an integral Hodge substructure of $H^k(X,\mathbb{Z})$.

One defines a bilinear form on $P^k(X,\mathbb{Z})$ by

$$Q_k(\alpha,\beta) = \int_X \alpha \wedge \beta \wedge \eta^{n-k}. \tag{2.3.6}$$

Definition 2.13. A *polarization* of an integral Hodge structure of weight k is a bilinear form

$$Q : H_{\mathbb{Z}} \times H_{\mathbb{Z}} \to \mathbb{Z} \tag{2.3.7}$$

which is nondegenerate over $\mathbb{Q}$ and satisfies the following conditions after extension to $H_{\mathbb{C}}$:

(1) $Q(u,v) = (-1)^k Q(v,u)$;
(2) $Q(H^{p,q}, H^{r,s}) = 0$ unless $(r,s) = (q,p)$;
(3) for $0 \neq v \in H^{p,q}$,

$$i^{p-q}(-1)^{k(k-1)/2} Q(v,\overline{v}) > 0. \tag{2.3.8}$$

Thus a polarization is alternating in odd weight and symmetric in even weight. The last two conditions are called the *Hodge–Riemann bilinear relations.*

Theorem 2.14 (Hodge–Riemann bilinear relations)**.** *The form Q_k polarizes the weight k Hodge structure on $P^k(X,\mathbb{Z})$. In particular, if $0 \neq \alpha \in P^k(X,\mathbb{C}) \cap H^{p,q}(X)$, then*

$$i^{p-q}(-1)^{k(k-1)/2} Q_k(\alpha,\overline{\alpha}) > 0. \tag{2.3.9}$$

Together, Theorems 2.11 and 2.14 are the Lefschetz theorems. A complete treatment of the analytic Hodge theorem, the Kähler identities, and the Lefschetz theorems may be found in [Voi02, Chapters 5–6].

2.4. **Curves.** Let C be a compact Riemann surface of genus g. We will reprove the Hodge decomposition theorem in this case. Holomorphic 1-forms are closed, so they define de Rham cohomology classes. Set

$$H^{1,0}(C) = H^0(C,\Omega^1_C) \subset H^1(C,\mathbb{C}), \qquad H^{0,1}(C) = \overline{H^{1,0}(C)}. \tag{2.4.1}$$

Proposition 2.15. *There is a direct-sum decomposition*

$$H^1(C,\mathbb{C}) = H^{1,0}(C) \oplus H^{0,1}(C). \tag{2.4.2}$$

The intersection pairing

$$Q(\alpha, \beta) = \int_C \alpha \wedge \beta \tag{2.4.3}$$

polarizes the resulting integral Hodge structure of weight 1 *on* $H^1(C, \mathbb{Z})$.

Proof. See Exercise 2.19. □

Choose a symplectic basis $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ of $H_1(C, \mathbb{Z})$ and normalize a basis $\omega_1, \ldots, \omega_g$ of $H^{1,0}(C)$ by

$$\int_{\alpha_i} \omega_j = \delta_{ij}. \tag{2.4.4}$$

The *period matrix*

$$\tau = \left(\int_{\beta_i} \omega_j \right)_{i,j} \tag{2.4.5}$$

is symmetric and has positive definite imaginary part; see Exercise 2.20. Thus τ belongs to the Siegel upper half space

$$\mathfrak{H}_g = \{\tau \in \mathrm{Mat}_{g \times g}(\mathbb{C}) : \tau^t = \tau,\ \operatorname{Im} \tau > 0\}. \tag{2.4.6}$$

For $g = 1$, this recovers the upper half plane and the period coordinate of Lecture 1. We will study $\mathfrak{H}_g$ in future lectures.

2.5. **Problem Set.**

Exercise 2.16 (The category of polarizable Hodge structures)**.** Develop the basics of the category of Hodge structures:

(1) Formulate the rational analogue of Definition 2.1. Then give the natural definition of a morphism between two rational Hodge structures of the same weight, both in terms of the Hodge decomposition and in terms of the Hodge filtration. Prove that the two formulations agree, and that the kernel and image of a morphism are Hodge substructures.

(2) Formulate the definition of a *polarizable* rational Hodge structure. What should the morphisms be in the category of polarizable Hodge structures?

(3) Prove that the category of polarizable rational Hodge structures of a fixed weight is semisimple: every Hodge substructure $V_{\mathbb{Q}} \subset H_{\mathbb{Q}}$ admits a complementary Hodge substructure. Deduce that every short exact sequence of polarizable rational Hodge structures of the same weight splits.

Exercise 2.17 (The Hodge diamond of a cubic threefold)**.** Let $X \subset \mathbb{P}^4$ be a smooth cubic threefold. Prove that $h^{2,1}(X) = 5$. The normal bundle sequence of X is

$$0 \to T_X \to T_{\mathbb{P}^4}|_X \to N_{X/\mathbb{P}^4} \simeq \mathcal{O}_X(3) \to 0.$$

Dualizing, it gives the conormal sequence

$$0 \to \mathcal{O}_X(-3) \to \Omega^1_{\mathbb{P}^4}|_X \to \Omega^1_X \to 0.$$

We also have on $\mathbb{P}^4$ the Euler sequence

$$0 \to \Omega^1_{\mathbb{P}^4} \to \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 5} \to \mathcal{O}_{\mathbb{P}^4} \to 0$$

and the restriction sequences

$$0 \to \mathcal{O}_{\mathbb{P}^4}(m-3) \to \mathcal{O}_{\mathbb{P}^4}(m) \to \mathcal{O}_X(m) \to 0$$

for all $m \in \mathbb{Z}$. Taking various long exact sequences in sheaf cohomology, use these to conclude that

$$H^1(X, \Omega^2_X) \simeq H^0(X, \mathcal{O}_X(1)) \simeq \mathbb{C}^5.$$

Exercise 2.18 (Primitive cohomology)**.** Let X be a smooth projective variety of dimension n, and let L be the Lefschetz operator associated to an ample class η.

(1) Prove that $P^k(X, \mathbb{Q})$ is a Hodge substructure of $H^k(X, \mathbb{Q})$.
(2) Derive the Lefschetz decomposition (2.3.5) from the Hard Lefschetz theorem.
(3) Suppose that X is a surface. Use the Hodge–Riemann bilinear relations to prove that the intersection form is negative definite on $P^2(X, \mathbb{R}) \cap H^{1,1}(X)$. Deduce that the intersection form on the Néron–Severi group $\mathrm{NS}(X) = \mathrm{im}(c_1\colon \mathrm{Pic}(X) \to H^2(X, \mathbb{Z}))$ signature $(1, r-1)$, $r = \operatorname{rank} \mathrm{NS}(X)$.

Exercise 2.19 (The polarized Hodge structure on $H^1(C, \mathbb{Z})$)**.** Complete the proof of Proposition 2.15. Let Q be the alternating, nondegenerate intersection form on $H^1(C, \mathbb{Z})$. First prove the orthogonality

$$Q(H^{1,0}(C), H^{1,0}(C)) = Q(H^{0,1}(C), H^{0,1}(C)) = 0$$

and, for every nonzero holomorphic 1-form ω, the positivity relation

$$iQ(\omega, \overline{\omega}) > 0.$$

For the latter, use the positivity of $\frac{i}{2}\omega \wedge \overline{\omega}$ discussed in Section 1.2.

Now prove the direct-sum decomposition (2.4.2). Riemann–Roch gives $\dim H^{1,0}(C) = g$, while $\dim H^1(C, \mathbb{C}) = 2g$, so it is enough to prove that $H^{1,0}(C) \cap H^{0,1}(C) = 0$. Conclude that Q satisfies the conditions in Definition 2.13.

Exercise 2.20 (The period matrix of a curve)**.** Let C be a genus g compact Riemann surface. Choose a symplectic basis $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$ of $H_1(C, \mathbb{Z})$. You may use the identity

$$\int_C \eta \wedge \xi = \sum_{i=1}^{g} \left(\int_{\alpha_i} \eta \int_{\beta_i} \xi - \int_{\beta_i} \eta \int_{\alpha_i} \xi \right) \tag{2.5.1}$$

for closed 1-forms η and ξ.

(1) Prove that the map

$$H^{1,0}(C) \to \mathbb{C}^g, \tag{2.5.2}$$

$$\omega \mapsto \left(\int_{\alpha_1} \omega, \dots, \int_{\alpha_g} \omega \right) \tag{2.5.3}$$

is an isomorphism. Use the positivity in Proposition 2.15 to prove injectivity.

(2) Let $\omega_1, \dots, \omega_g$ be the normalized basis (2.4.4). Use the orthogonality and positivity of the polarization, together with (2.5.1), to prove that the period matrix $\tau = (\int_{\beta_i} \omega_j)$ satisfies

$$\tau^t = \tau, \qquad \operatorname{Im} \tau > 0. \tag{2.5.4}$$

(3) Show that the complex torus

$$\operatorname{Jac}(C) = \mathbb{C}^g / (\mathbb{Z}^g \oplus \tau \mathbb{Z}^g) \tag{2.5.5}$$

depends only on the polarized Hodge structure on $H^1(C, \mathbb{Z})$. When $g = 1$, compare this construction with $E_\tau = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$ from Lecture 1.

Exercise 2.21 (A product of curves)**.** Let C_1 and C_2 be compact Riemann surfaces of genera g_1 and g_2. Use the Künneth isomorphism

$$H^2(C_1 \times C_2, \mathbb{C}) \simeq H^2(C_1, \mathbb{C}) \oplus \big(H^1(C_1, \mathbb{C}) \otimes H^1(C_2, \mathbb{C})\big) \oplus H^2(C_2, \mathbb{C})$$

to determine $H^{2,0}$, $H^{1,1}$, and $H^{0,2}$ and their dimensions. Describe the resulting Hodge filtration on $H^2(C_1 \times C_2, \mathbb{C})$.

3. Lecture 3: VHS and period maps

3.1. **Local systems and flat connections.** Throughout this lecture, B is a connected complex manifold.

Definition 3.1. A *local system* on B is a locally constant sheaf $\mathbb{V}$: every point of B has a connected open neighborhood U for which

$$\mathbb{V}|_U \simeq \underline{V}_U \tag{3.1.1}$$

for some set V, where $\underline{V}_U$ is the constant sheaf on U with fiber V. An A-*local system* means a local system of A-modules; usually we will take $A = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, or $\mathbb{C}$.

If the stalks of a local system of A-modules have dimension r, we call r the *rank* of $\mathbb{V}$.

Example 3.2 (The orientation local system)**.** The definition makes sense on any topological space. Let M be a Möbius strip. The local choices of an orientation of M form a local system $\mathbb{V}$ whose stalk has two elements, identified with $\mathbb{Z}/2\mathbb{Z}$ after choosing one of them. Continuation once around the core circle interchanges the two orientations. Thus $\mathbb{V}$ is locally constant, but not constant on M; for instance, it admits no global section.

Sections of a local system can be continued along paths. After choosing a base point $b \in B$, continuation around loops gives the *monodromy representation*

$$\rho : \pi_1(B, b) \to \mathrm{Aut}(\mathbb{V}_b). \tag{3.1.2}$$

Conversely, a representation of $\pi_1(B, b)$ determines a local system; see Exercise 3.24.

Definition 3.3. Let $\mathcal{V}$ be a holomorphic vector bundle on B. A *holomorphic connection* on $\mathcal{V}$ is a $\mathbb{C}$-linear map

$$\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega^1_B \tag{3.1.3}$$

satisfying the Leibniz rule

$$\nabla(fs) = s \otimes df + f\nabla s.$$

The connection ∇ induces a natural $\mathbb{C}$-linear map $\mathcal{V} \otimes \Omega^k_B \to \mathcal{V} \otimes \Omega^{k+1}_B$ for all $k \geq 0$. The connection is *flat* if its curvature $\nabla \circ \nabla$ vanishes, see Exercise 3.28.

A local system $\mathbb{V}$ of complex vector spaces gives the holomorphic vector bundle

$$\mathcal{V} = \mathbb{V} \otimes_{\mathbb{C}} \mathcal{O}_B. \tag{3.1.4}$$

On an open set where $\mathbb{V}$ is constant, define

$$\nabla(v \otimes f) = v \otimes df. \tag{3.1.5}$$

These local formulas agree on overlaps and define a flat connection. Its locally constant sections are precisely the sections of $\mathbb{V}$.

Proposition 3.4. *The constructions*

$$\mathbb{V} \mapsto (\mathbb{V} \otimes \mathcal{O}_B, \nabla),$$
$$(\mathcal{V}, \nabla) \mapsto \ker(\nabla)$$

give an equivalence between complex local systems and holomorphic vector bundles with flat holomorphic connection.

3.2. **The Gauss–Manin connection.** Let $\pi : \mathcal{X} \to B$ be a proper holomorphic submersion, and write $X_b = \pi^{-1}(b)$ for the fiber over $b \in B$. The following theorem explains why the cohomology groups of the fibers define a local system.

Theorem 3.5 (Ehresmann)**.** *A proper submersion of smooth manifolds is locally a C^∞ fiber bundle.*

Thus, there is an open neighborhood U about any point $b \in B$, for which $\mathcal{X}|_U \simeq_{\text{difeo}} X_b \times U$ are diffeomorphic over U. In particular,

$$\mathbb{H} = R^k \pi_* \mathbb{Z} \tag{3.2.1}$$

is a local system with stalk $H^k(X_b, \mathbb{Z})$ at $b \in B$. Its associated holomorphic vector bundle

$$\mathcal{H} = \mathbb{H} \otimes_{\mathbb{Z}} \mathcal{O}_B \tag{3.2.2}$$

carries the flat connection supplied by (3.1.5).

Definition 3.6. The flat connection

$$\nabla : \mathcal{H} \to \mathcal{H} \otimes \Omega^1_B$$

obtained from the Betti local system $R^k\pi_*\mathbb{Z}$ is called the *Gauss–Manin connection.*

Suppose now that $\pi : \mathcal{X} \to B$ is a smooth projective family. For each $b \in B$, the Hodge decomposition of X_b gives a filtration

$$F^p H^k(X_b, \mathbb{C}) \subset H^k(X_b, \mathbb{C}) = \mathcal{H}_b.$$

The following theorem describes how these subspaces vary with b.

Theorem 3.7 (Griffiths holomorphicity and transversality)**.** *Let $\pi : \mathcal{X} \to B$ be a smooth projective family.*

(1) *The fiberwise Hodge filtrations form holomorphic subbundles*

$$F^p\mathcal{H} \subset \mathcal{H}, \qquad (F^p\mathcal{H})_b = F^p H^k(X_b, \mathbb{C}). \tag{3.2.3}$$

(2) *The Gauss–Manin connection lowers the Hodge filtration by at most one step:*

$$\nabla(F^p\mathcal{H}) \subset F^{p-1}\mathcal{H} \otimes \Omega^1_B. \tag{3.2.4}$$

Remark 3.8. For an algebraic smooth projective family, $\mathcal{H}$ and ∇ also admit a purely algebro-geometric construction using relative algebraic de Rham hypercohomology and the algebraic Gauss–Manin connection.

The Gauss–Manin connection, Griffiths transversality, and abstract variations of Hodge structure are treated systematically in [CMSP17, Chapters 4–5]; see also [Voi02, Part III].

3.3. **Polarized variations of Hodge structure.**

Definition 3.9. A *$\mathbb{Z}$-polarized variation of Hodge structure of weight* k, abbreviated $\mathbb{Z}$-PVHS, on B consists of the following data:

(1) a local system $\mathbb{V}$ of finitely generated abelian groups;
(2) a bilinear form, nondegenerate over $\mathbb{Q}$,
$$Q : \mathbb{V} \times \mathbb{V} \to \mathbb{Z}$$
which is $(-1)^k$-symmetric;
(3) a decreasing filtration $F^\bullet\mathcal{V}$ of $\mathcal{V} = \mathbb{V} \otimes_{\mathbb{Z}} \mathcal{O}_B$ by holomorphic subbundles,

such that

(A) for every $b \in B$, the filtration $F^\bullet_b$ defines a weight k Hodge structure on $\mathbb{V}_b$ polarized by Q_b and
(B) the filtration satisfies *Griffiths transversality*:
$$\nabla(F^p\mathcal{V}) \subset F^{p-1}\mathcal{V} \otimes \Omega^1_B, \tag{3.3.1}$$
where ∇ is the flat connection associated to $\mathbb{V}_{\mathbb{C}} = \mathbb{V} \otimes_{\mathbb{Z}} \mathbb{C}$.

Since $\mathbb{V}$ is a local system, the pairing Q is by definition locally constant. Equivalently,

$$dQ(s,t) = Q(\nabla s, t) + Q(s, \nabla t)$$

for local sections s, t of $\mathcal{V}$.

The basic source of $\mathbb{Z}$-PVHS is geometry. Let $\pi : \mathcal{X} \to B$ be a smooth projective family of relative dimension n, and let $\mathcal{L}$ be a relatively ample line bundle. Cup product with $c_1(\mathcal{L})$ gives a morphism of local systems, denoted by L. For $k \le n$, set

$$\mathbb{H}_{\mathrm{prim}} = \ker\left(L^{n-k+1} : R^k\pi_*\mathbb{Z} \to R^{2n-k+2}\pi_*\mathbb{Z}\right). \tag{3.3.2}$$

Theorem 3.10 (Geometric $\mathbb{Z}$-PVHS)**.** *The primitive local system $\mathbb{H}_{\mathrm{prim}}$ with its Hodge filtration and the fiberwise form Q_k of* (2.3.6) *define a $\mathbb{Z}$-PVHS of weight k. The Hodge–Riemann bilinear relations hold fiberwise and Griffiths transversality holds in the family.*

For a family of smooth projective curves, H^1 is already primitive. Hence $R^1\pi_*\mathbb{Z}$ with its intersection form is a weight one $\mathbb{Z}$-PVHS.

3.4. **Weight one Hodge structures and Siegel space.** We now describe the period domain for the weight one Hodge structures arising from families of curves. We begin by fixing their underlying lattice and polarization.

Definition 3.11 (The standard symplectic lattice)**.** Let $H_{\mathbb{Z}} = \mathbb{Z}^{2g}$ have basis

$$e_1, \dots, e_g, f_1, \dots, f_g.$$

Define an alternating form Q by

$$Q(e_i, e_j) = Q(f_i, f_j) = 0, \qquad Q(e_i, f_j) = -\delta_{ij}. \tag{3.4.1}$$

If $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$ is a symplectic basis of the homology of a curve and α_i^*, β_i^* is its dual cohomology basis, our convention is

$$f_i = \alpha_i^*, \qquad e_i = \beta_i^*. \tag{3.4.2}$$

Thus the sign in (3.4.1) agrees with the intersection pairing on cohomology.

A weight 1 Hodge structure on $H_{\mathbb{Z}}$ is determined by the single subspace $F^1 = H^{1,0} \subset H_{\mathbb{C}}$. For a curve C, if the basis $\omega_1, \dots, \omega_g$ of $H^{1,0}(C)$ is normalized by the condition

$$\int_{\alpha_i} \omega_j = \delta_{ij},$$

then

$$[\omega_j] = f_j + \sum_{i=1}^{g} \left(\int_{\beta_i} \omega_j \right) e_i. \tag{3.4.3}$$

Thus the period matrix records $H^{1,0}(C)$ in the basis of Definition 3.11. Note that $\tau = (\int_{\beta_i} \omega_j)_{i,j}$ is a symmetric matrix, with positive definite imaginary part by Exercise 2.20.

Definition 3.12 (The weight one period domain)**.** The *period domain of polarized weight one Hodge structures* on $(H_{\mathbb{Z}}, Q)$, with Hodge numbers (g, g), is the set of g-dimensional subspaces $F^1 \subset H_{\mathbb{C}}$ such that

$$Q(F^1, F^1) = 0, \qquad H_{\mathbb{C}} = F^1 \oplus \overline{F^1}, \qquad iQ(v, \overline{v}) > 0 \quad (0 \neq v \in F^1). \tag{3.4.4}$$

Proposition 3.13. *The period domain of polarized weight one Hodge structures on* $(H_{\mathbb{Z}}, Q)$ *is naturally identified with the Siegel space* $\mathfrak{H}_g$*. Under this identification,*

$$\tau \in \mathfrak{H}_g \leftrightarrow F^1_\tau = \mathrm{Span}_{\mathbb{C}} \{f_j + \textstyle\sum_{i=1}^g \tau_{ij} e_i : j = 1, \dots, g\} \subset H_{\mathbb{C}}. \quad (3.4.5)$$

Proof. The symmetry and positivity of τ imply the first and third conditions in (3.4.4); positive-definiteness of $\mathrm{Im}\,\tau$ and an analysis of dimensions give the second condition.

Conversely, positivity in (3.4.4) ensures that the projection of F^1_τ onto the span of $f_1, \dots, f_g$ an isomorphism. Thus, we may write F^1_τ as a graph of a morphism from the span of $f_1, \dots, f_g$ to the span of the $e_1, \dots, e_g$. Isotropy and positivity ensure that this morphism is given by a unique symmetric matrix τ with $\mathrm{Im}\,\tau > 0$. □

Definition 3.14. For a commutative ring A, the *symplectic group* of $(H_{\mathbb{Z}}, Q)$ over A is

$$\mathrm{Sp}_{2g}(A) = \{\gamma \in \mathrm{GL}(H_{\mathbb{Z}} \otimes_{\mathbb{Z}} A) : Q(\gamma u, \gamma v) = Q(u, v)\}, \quad (3.4.6)$$

where Q is extended A-bilinearly.

Proposition 3.15. *The real group* $\mathrm{Sp}_{2g}(\mathbb{R})$ *acts transitively on* $\mathfrak{H}_g$*, and the stabilizer of* iI_g *is* $U(g)$*. Therefore*

$$\mathfrak{H}_g \simeq \mathrm{Sp}_{2g}(\mathbb{R})/U(g). \quad (3.4.7)$$

If we write in 2×2 *block form*

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbb{Z}),$$

then the natural (left) action on $\mathfrak{H}_g \simeq \mathrm{Sp}_{2g}(\mathbb{R})/U(g)$ *is*

$$\gamma \cdot \tau = (A\tau + B)(C\tau + D)^{-1}. \quad (3.4.8)$$

Proof. The first part can be proven by choosing an orthonormal basis of F^1 for the Hermitian form $iQ(v, \bar{v})$. The group $\mathrm{Sp}_{2g}(\mathbb{R})$ acts freely transitively on the set of all such bases of all possible F^1. The subgroup stabilizing such a hermitian orthonormal basis is $U(g)$.

For the action formula, write the spanning vectors in (3.4.5) of F^1 as the columns of the $2g \times g$ matrix

$$\begin{pmatrix} \tau \\ I_g \end{pmatrix}.$$

Applying γ gives

$$\begin{pmatrix} A\tau + B \\ C\tau + D \end{pmatrix}.$$

To restore the normalization of the α-periods (recall that the bottom $g\times g$ block expresses the coefficients of the α_i^*), we multiply on the right by $(C\tau+D)^{-1}$. The upper block is then $(A\tau+B)(C\tau+D)^{-1}$, which proves (3.4.8). When $g=1$, this is the fractional linear transformation

$$\tau \longmapsto \frac{a\tau+b}{c\tau+d}$$

from the first lecture: one divides by the new α-period in order to renormalize it to equal 1. □

The description as a homogenous space is analyzed in the more general setting of Exercise 3.29.

3.5. **Abelian varieties and $\mathcal{A}_g$.**

Definition 3.16. A *complex abelian variety* is a projective complex torus $(A,0)$.

A *principal polarization* on an abelian variety $(A,0)$ is an ample class $L\in H^2(A,\mathbb{Z})\cap H^{1,1}(A)$ (i.e. the class $L=c_1(\mathcal{L})$ of an ample line bundle), which defines a unimodular alternating form on $H_1(A,\mathbb{Z})$, via the isomorphism $H^2(A,\mathbb{Z})\simeq \wedge^2 H_1(A,\mathbb{Z})^\vee$.

A *principally polarized abelian variety* (or PPAV) $(A,0,L)$ consists of an abelian variety and a principal polarization.

A matrix $\tau\in\mathfrak{H}_g$ also defines a complex torus

$$A_\tau=\mathbb{C}^g/(\mathbb{Z}^g\oplus\tau\mathbb{Z}^g), \tag{3.5.1}$$

regardless of whether τ is the period matrix of a genus g Riemann surface. We have:

Fact 3.17. The symmetry of τ and the positivity of $\operatorname{Im}\tau$ imply that the standard symplectic form on $H_1(A_\tau,\mathbb{Z})=\operatorname{Span}_\mathbb{Z}\{\alpha_i,\beta_i\,:\,i=1,\dots,g\}$ is the first Chern class of a principal polarization; in particular, A_τ is projective. Conversely, after choosing a symplectic basis, the periods of a principally polarized abelian variety can be normalized to the form (τ, I_g) with $\tau\in\mathfrak{H}_g$.

Proposition 3.18. *Every principally polarized complex abelian variety of dimension g is isomorphic to A_τ for some $\tau\in\mathfrak{H}_g$.*

Moreover, A_τ and $A_{\tau'}$ are isomorphic as principally polarized abelian varieties if and only if

$$\tau'=\gamma\cdot\tau$$

for some $\gamma\in\mathrm{Sp}_{2g}(\mathbb{Z})$.

For complex tori, the Riemann bilinear relations, and the analytic construction of $\mathcal{A}_g$, see [BL04, Chapters 1, 4, and 8].

Definition 3.19. The *Siegel modular variety of genus g* is

$$\mathcal{A}_g = \mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathfrak{H}_g. \tag{3.5.2}$$

By Proposition 3.18, its points parameterize isomorphism classes of principally polarized abelian varieties of dimension g.

For $g = 1$, this recovers the quotient $\mathcal{A}_1$ of Lecture 1.

Over $\mathfrak{H}_g$ the universal family can be written explicitly as

$$\mathcal{X}_g^{\mathrm{univ}} = \mathbb{Z}^{2g}\backslash(\mathfrak{H}_g \times \mathbb{C}^g), \tag{3.5.3}$$

where

$$(m, n) \cdot (\tau, z) = (\tau, z + m + \tau n). \tag{3.5.4}$$

Writing γ as in (3.4.8), the symplectic group acts by

$$\gamma \cdot (\tau, z) = \big(\gamma \cdot \tau, (C\tau + D)^{-t} z\big)\,. \tag{3.5.5}$$

Together these actions give

$$\mathcal{X}_g := \big(\mathrm{Sp}_{2g}(\mathbb{Z}) \ltimes \mathbb{Z}^{2g}\big)\backslash(\mathfrak{H}_g \times \mathbb{C}^g) \to \mathcal{A}_g \tag{3.5.6}$$

but as in Warning 1.16, automorphisms prevent this ordinary quotient from being a universal family over the coarse quotient.

3.6. **Period maps and the Torelli theorems.** We first describe the space in which a period map takes values. Fix a lattice $H_{\mathbb{Z}}$, a nondegenerate $\mathbb{Z}$-valued form Q which is $(-1)^k$-symmetric, and Hodge numbers

$$h^{p,q} = h^{q,p}, \qquad p + q = k.$$

Put $f^p = \sum_{r \geq p} h^{r,k-r}$ and $H_{\mathbb{C}} = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$.

Definition 3.20 (The compact dual and period domain)**.** The *compact dual* $\check{D}$ is a connected component of the space of flags

$$\cdots \supset F^p \supset F^{p+1} \supset \cdots$$

satisfying $F^p = H_{\mathbb{C}}$ for $p \ll 0$, $F^p = 0$ for $p \gg 0$, and

$$\dim F^p = f^p, \qquad Q(F^p, F^{k-p+1}) = 0. \tag{3.6.1}$$

The *period domain* $D \subset \check{D}$ consists of the flags for which

$$H_{\mathbb{C}} = F^p \oplus \overline{F^{k-p+1}} \qquad \text{for every } p, \tag{3.6.2}$$

and for which the spaces

$$H^{p,q} = F^p \cap \overline{F^q}$$

satisfy the positivity condition (2.3.8). Thus D parameterizes polarized Hodge structures on the fixed lattice $(H_{\mathbb{Z}}, Q)$ with Hodge numbers $h^{p,q}$.

The conditions defining D inside $\check{D}$ are open. The flag variety $\check{D}$ and the geometry of this open subset are analyzed in Exercise 3.29.

The tangent space to the Grassmannian at a subspace $W \subset V$ is $\mathrm{Hom}(W, V/W)$. Thus, a first-order deformation of a flag $F^\bullet$ determines, for every p, a linear map

$$\xi_p : F^p \to H_{\mathbb{C}}/F^p.$$

Definition 3.21 (The horizontal subbundle)**.** A tangent vector $\xi \in T_{[F^\bullet]}D$ is *horizontal* if

$$\xi_p(F^p) \subset F^{p-1}/F^p \qquad \text{for every } p. \tag{3.6.3}$$

The horizontal tangent spaces form a holomorphic subbundle

$$T_D^{\mathrm{hor}} \subset T_D.$$

Definition 3.22. A holomorphic map $\Psi : S \to D$ is *horizontal* if

$$d\Psi(T_S) \subset T_D^{\mathrm{hor}}.$$

Now let $(\mathbb{V}, Q, F^\bullet)$ be a $\mathbb{Z}$-PVHS on B, fix $b_0 \in B$, and set $H_{\mathbb{Z}} = \mathbb{V}_{b_0}$. Its weight and Hodge numbers determine the period domain D of Definition 3.20. Parallel transport on the universal cover identifies every fiber of $\mathbb{V}$ with $H_{\mathbb{Z}}$. The Hodge filtration in each fiber therefore gives a map

$$\widetilde{\Phi} : \widetilde{B} \to D. \tag{3.6.4}$$

Griffiths holomorphicity says that $\widetilde{\Phi}$ is holomorphic. Griffiths transversality says precisely that its derivative satisfies (3.6.3). Hence $\widetilde{\Phi}$ is horizontal.

Continuation around loops gives the monodromy representation

$$\rho : \pi_1(B, b_0) \to \mathrm{Aut}(H_{\mathbb{Z}}, Q), \tag{3.6.5}$$

and (3.6.4) is ρ-equivariant. It therefore descends to the *period map*

$$\Phi : B \to \rho(\pi_1(B, b_0))\backslash D. \tag{3.6.6}$$

The map Φ is called holomorphic and horizontal because its local lifts to D have these properties. Thus Griffiths holomorphicity and transversality are exactly the two statements which make the period map a holomorphic horizontal map.

Let $\pi : \mathcal{E} \to B$ be a family of elliptic curves. On a simply connected open set $U \subset B$, choose a flat symplectic basis α, β of $R^1\pi_*\mathbb{Z}$ and a relative holomorphic 1-form ω normalized by

$$\int_\alpha \omega = 1.$$

The local expression for the period map is

$$\tau(b) = \int_{\beta_b} \omega_b \in \mathfrak{H}. \tag{3.6.7}$$

Changing the symplectic basis acts on τ by $\mathrm{SL}_2(\mathbb{Z})$, so these local maps combine to give

$$B \to \mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H} = \mathcal{A}_1.$$

Now let $\pi : \mathcal{C} \to B$ be a family of genus g curves. Choose locally a flat symplectic basis $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$. There is a unique basis $\omega_1, \dots, \omega_g$ of relative holomorphic 1-forms normalized by

$$\int_{\alpha_i} \omega_j = \delta_{ij};$$

in other words, its α-period matrix is I_g. The remaining periods form the matrix

$$\tau(b) = \left(\int_{\beta_i} \omega_j \right)_{i,j} \in \mathfrak{H}_g. \tag{3.6.8}$$

Since a change of symplectic basis acts by (3.4.8), these matrices define the period map

$$B \to \mathcal{A}_g, \qquad b \mapsto [\mathrm{Jac}(C_b)]. \tag{3.6.9}$$

Theorem 3.23 (Torelli)**.** *Let C and C' be compact Riemann surfaces of genus $g \geq 1$. Then $C \simeq C'$ if and only if their principally polarized Jacobians are isomorphic. Equivalently, the polarized integral Hodge structure on $H^1(C, \mathbb{Z})$ determines C up to isomorphism.*

For the construction and geometry of Jacobians, including the Torelli theorem, see [BL04, Chapter 11].

For $g = 1$, the Jacobian of an elliptic curve is the curve itself. Thus

$$\mathcal{A}_1 = \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$$

is the first Torelli theorem: the periods, modulo change of symplectic basis, recover the elliptic curve.

3.7. Problem Set.

Exercise 3.24 (Local systems and flat bundles)**.** Let B be a manifold.

(1) Starting from a local system $\mathbb{V}$, prove that continuation of sections along paths gives a monodromy representation (3.1.2).

(2) Conversely, let V be an A-module and $\rho : \pi_1(B, b) \to \mathrm{Aut}_A(V)$ a representation. Construct the corresponding local system $\mathbb{V}$ from the universal cover of B, and prove that these two constructions are inverse up to isomorphism.

(3) Prove Proposition 3.4. In the reverse direction, use flatness to show that a flat section through any vector in one fiber exists locally and is unique.

Exercise 3.25 (The universal abelian family)**.** Let $\gamma \in \mathrm{Sp}_{2g}(\mathbb{Z})$ have blocks A, B, C, D.

(1) Prove that (3.4.8) preserves $\mathfrak{H}_g$.

(2) Show that the linear map

$$z \mapsto (C\tau + D)^{-t} z$$

carries the lattice $\mathbb{Z}^g \oplus \tau\mathbb{Z}^g$ isomorphically onto $\mathbb{Z}^g \oplus (\gamma \cdot \tau)\mathbb{Z}^g$. Deduce that (3.5.5) is compatible with (3.5.4).

Exercise 3.26 (The period map of a pencil of cubic curves)**.** Let

$$V = H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3)), \qquad \mathbb{P}(V) \simeq \mathbb{P}^9,$$

and let $\Delta \subset \mathbb{P}(V)$ be the discriminant hypersurface parameterizing singular plane cubics.

(1) Choose a general pencil $\ell \simeq \mathbb{P}^1 \subset \mathbb{P}(V)$, spanned by cubics F_0 and F_1. In homogeneous coordinates $[x_0 : x_1 : x_2]$ on $\mathbb{P}^2$ and $[s : t]$ on ℓ, consider

$$\mathcal{F}(x, s, t) = sF_0(x) + tF_1(x).$$

Show that the common zero locus in $\mathbb{P}^2 \times \ell$ of

$$\frac{\partial \mathcal{F}}{\partial x_0}, \qquad \frac{\partial \mathcal{F}}{\partial x_1}, \qquad \frac{\partial \mathcal{F}}{\partial x_2}$$

parameterizes the singular points of the members of the pencil. Each equation has bidegree $(2, 1)$. If h and u are the hyperplane classes of $\mathbb{P}^2$ and ℓ, compute

$$(2h + u)^3$$

and deduce that $\deg \Delta = 12$. Why does a general pencil have twelve singular members, each an irreducible cubic with one exactly node?

(2) A general pencil has nine distinct base points. Blow them up to obtain

$$\pi : X \to \ell,$$

and choose one exceptional curve as the zero section. Over the complement U of the twelve singular fibers, this is a family of elliptic curves and hence has a period map

$$\Phi : U \to \mathcal{A}_1.$$

Using Exercise 1.24, show that Φ extends to a map

$$j_\ell : \ell \to X(1) \simeq \mathbb{P}^1_j. \tag{3.7.1}$$

Determine the behavior of the map j_ℓ at the twelve points corresponding to singular fibers and conclude that

$$\deg(j_\ell) = 12.$$

Exercise 3.27 (The unipotent quotient and a Siegel modular form)**.** The definitions and theta series construction in this exercise may be compared with [Kli90, Chapters 1 and 8]. Let $k \geq 0$. A *Siegel modular form of genus* 2 *and weight* k is a holomorphic function $F : \mathfrak{H}_2 \to \mathbb{C}$ satisfying

$$F(\gamma \cdot \tau) = \det(C\tau + D)^k F(\tau) \tag{3.7.2}$$

for every

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{Z}),$$

and which is holomorphic at the cusps. Part (2) makes the latter condition explicit.

Write a point of $\mathfrak{H}_2$ as

$$\tau = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix}, \qquad \tau_{12} = \tau_{21}, \tag{3.7.3}$$

and consider the subgroup

$$U(\mathbb{Z}) = \left\{ \begin{pmatrix} I_2 & B \\ 0 & I_2 \end{pmatrix} : B \in \mathrm{Sym}_2(\mathbb{Z}) \right\} \subset \mathrm{Sp}_4(\mathbb{Z}). \tag{3.7.4}$$

(1) Show that $U(\mathbb{Z})$ acts by $\tau \mapsto \tau + B$ and that

$$\begin{aligned} &U(\mathbb{Z})\backslash\mathfrak{H}_2 \to (\mathbb{C}^*)^3, \\ &\qquad [\tau] \mapsto (q_{11}, q_{12}, q_{22}) = \left(e^{2\pi i \tau_{11}}, e^{2\pi i \tau_{12}}, e^{2\pi i \tau_{22}}\right) \end{aligned} \tag{3.7.5}$$

is an open embedding. Prove that its image is

$$\left\{ (q_{11}, q_{12}, q_{22}) \in (\mathbb{C}^*)^3 : \begin{array}{c} |q_{11}|, |q_{22}| < 1, \\ \log|q_{11}| \log|q_{22}| > (\log|q_{12}|)^2 \end{array} \right\}. \tag{3.7.6}$$

(2) The transformation law (3.7.2) makes F invariant under $U(\mathbb{Z})$. Deduce that it has a Fourier expansion

$$F(\tau) = \sum_T a(T) e^{2\pi i \,\mathrm{tr}(T\tau)}, \tag{3.7.7}$$

where T ranges over symmetric, half-integral 2×2 matrices: $2T$ is integral and has even diagonal. For

$$T = \begin{pmatrix} n & \ell/2 \\ \ell/2 & m \end{pmatrix},$$

show that

$$e^{2\pi i \operatorname{tr}(T\tau)} = q_{11}^n q_{12}^\ell q_{22}^m, \tag{3.7.8}$$

and that $T \geq 0$ is equivalent to

$$n, m \geq 0, \qquad 4nm - \ell^2 \geq 0.$$

We say that $F(\tau)$ is *holomorphic at the standard cusp of $\mathcal{A}_2$* if $a(T) = 0$ unless $T \geq 0$.

(3) Let $(L, \cdot)$ be a positive definite, even unimodular lattice of rank N. Define its genus-two theta series by

$$\begin{aligned}\Theta_L^{(2)}(\tau) &= \sum_{x,y\in L} \exp\left(\pi i((x\cdot x)\tau_{11} + 2(x\cdot y)\tau_{12} + (y\cdot y)\tau_{22})\right) \\ &= \sum_{x,y\in L} e^{2\pi i \operatorname{tr}(T(x,y)\tau)}, \qquad T(x,y) = \frac{1}{2}\begin{pmatrix} x\cdot x & x\cdot y \\ y\cdot x & y\cdot y \end{pmatrix}.\end{aligned} \tag{3.7.9}$$

Prove that the series converges locally uniformly on $\mathfrak{H}_2$. Show that every $T(x,y)$ is positive semidefinite and half-integral, so (3.7.9) has the form required by (3.7.7).

(4) Prove that $\Theta_L^{(2)}$ is a Siegel modular form of weight $N/2$. It is enough to check the transformation law on the following generators of $\mathrm{Sp}_4(\mathbb{Z})$:

$$\begin{pmatrix} I_2 & B \\ 0 & I_2 \end{pmatrix}, \qquad \begin{pmatrix} A & 0 \\ 0 & A^{-t} \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix}, \tag{3.7.10}$$

where $B \in \mathrm{Sym}_2(\mathbb{Z})$ and $A \in \mathrm{GL}_2(\mathbb{Z})$. For the first two types, use evenness and reindex the summation. For S, apply Poisson summation to $L \oplus L$ and use unimodularity to obtain

$$\Theta_L^{(2)}(-\tau^{-1}) = \det(-i\tau)^{N/2}\Theta_L^{(2)}(\tau). \tag{3.7.11}$$

Recall that the rank of an even unimodular lattice is divisible by 8, and conclude that the factor in (3.7.11) is $\det(\tau)^{N/2}$. In particular, the E_8 lattice gives a genus-two Siegel modular form of weight 4. Check that

$$\Theta_{E_8}^{(2)}\begin{pmatrix} \tau_{11} & 0 \\ 0 & \tau_{22} \end{pmatrix} = E_4(\tau_{11})E_4(\tau_{22}). \tag{3.7.12}$$

Exercise 3.28 (The curvature of a connection)**.** Let $\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega^1_B$ be a holomorphic connection.

(1) Find a formula extending ∇ to maps

$$\nabla : \mathcal{V} \otimes \Omega^k_B \to \mathcal{V} \otimes \Omega^{k+1}_B \tag{3.7.13}$$

which agree with the original connection when $k = 0$ and satisfy the graded Leibniz rule

$$\nabla(\xi \wedge \beta) = \nabla\xi \wedge \beta + (-1)^k \xi \wedge d\beta$$

for $\xi \in \mathcal{V} \otimes \Omega^k_B$ and $\beta \in \Omega^\ell_B$. Prove that your formula is well-defined and is the unique such extension.

(2) Use your formula to prove that

$$\nabla^2(fs) = f\nabla^2(s)$$

for every local holomorphic function f and local section s of $\mathcal{V}$. Conclude that the curvature is an $\mathcal{O}_B$-linear map

$$F_\nabla = \nabla^2 : \mathcal{V} \to \mathcal{V} \otimes \Omega^2_B, \tag{3.7.14}$$

or equivalently defines a global section of

$$\mathrm{End}(\mathcal{V}) \otimes \Omega^2_B.$$

In a local frame, write $\nabla = d + A$ with $A \in \mathrm{End}(\mathcal{V}) \otimes \Omega^1_B$ and prove the formula

$$F_\nabla = dA + A \wedge A.$$

Exercise 3.29 (The geometry of a period domain)**.** For a systematic account of the homogeneous geometry and horizontal tangent bundle of a period domain, see [CMSP17, Chapter 12]. Use the notation of Definition 3.20, and set

$$G_\mathbb{C} = \mathrm{Aut}(H_\mathbb{C}, Q), \qquad G_\mathbb{R} = \mathrm{Aut}(H_\mathbb{R}, Q).$$

(1) Show that $\check{D}$ is a complex flag variety, homogeneous under $G_\mathbb{C}$. Identify the stabilizer of a flag as a parabolic subgroup. Then prove that the conditions (3.6.2) and (2.3.8) define an open subset $D \subset \check{D}$, and that $G_\mathbb{R}$ acts transitively on each connected component of D.

(2) Fix a connected component of D and a point in that component. Compute its stabilizer $K \subset G_\mathbb{R}$. Show, up to passing to identity

components, that this component is isomorphic to $G_{\mathbb{R}}/K$ and

$$K \simeq \begin{cases} \prod_{p>q} U(h^{p,q}), & k \text{ odd}, \\ O(h^{k/2,k/2}) \times \prod_{p>q} U(h^{p,q}), & k \text{ even}. \end{cases} \tag{3.7.15}$$

In particular, K is compact. Identify $G_{\mathbb{R}}$ as a real symplectic group in odd weight and as a real orthogonal group in even weight.

(3) Recall that $T_{[W]} \operatorname{Gr}(r, V) = \operatorname{Hom}(W, V/W)$. Use the embedding of $\check{D}$ into a product of Grassmannians to describe $T_{[F^\bullet]}\check{D}$ as compatible collections of maps

$$F^p \to H_{\mathbb{C}}/F^p$$

satisfying the linearization of (3.6.1). Prove that the condition (3.6.3) defines a holomorphic subbundle of T_D. In terms of the Hodge decomposition, identify its fiber with the subspace of

$$\bigoplus_{p+q=k} \operatorname{Hom}(H^{p,q}, H^{p-1,q+1})$$

whose elements are infinitesimal isometries of Q.

Exercise 3.30 (When Griffiths transversality is vacuous)**.** In each of the following cases, prove that every tangent vector to the period domain is horizontal.

(1) For the weight one period domain $\mathfrak{H}_g \simeq \operatorname{Sp}_{2g}(\mathbb{R})/U(g)$, observe that the only nontrivial part of the filtration is $F^1 \subset F^0 = H_{\mathbb{C}}$ and interpret (3.6.3).

(2) Let $(H_{\mathbb{Z}}, Q)$ be a symmetric lattice of signature $(2, n)$ and consider weight two polarized $\mathbb{Z}$-Hodge structures on it with Hodge numbers $(1, n, 1)$. Show that the corresponding period domain is isomorphic to an open subset of a quadric:

$$\{[\omega] \in \mathbb{P}(H_{\mathbb{C}}) : Q(\omega, \omega) = 0,\ Q(\omega, \overline{\omega}) > 0\}. \tag{3.7.16}$$

Differentiate the equation $Q(\omega, \omega) = 0$ along a first order perturbation of ω to prove that a tangent vector sends F^2 into F^1/F^2. Conclude that transversality is automatic for this $O(2, n)$ period domain as well.

For further reading on variations of Hodge structure and period domains, see [Voi02, CMSP17]; for complex abelian varieties and Siegel modular forms, see [BL04, Kli90].

4. Lecture 4: Catching up

We will finish up whatever material we didn't manage to cover in the previous lectures. If time permits, we will discuss some further advanced topics, such as the Baily–Borel compactification.

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